

# Principles of Distributed Computing

## Exercise 5: Sample Solution

### 1 Pancake Networks

Generally, observe that  $N = |V(P_n)| = n! \in O(n^n) \Rightarrow n \in O(\frac{\log N}{\log \log N})$ .

- a) See Figure 1. For drawing  $P_n$ , first draw  $n$  copies of  $P_{n-1}$ , each of which will have some  $j \in [n]$  fixed as the last vertex. Then there are  $(n - 2)!$  nodes of such a  $P_{n-1}$  connected to the same  $(n - 1)$ -dimensional pancake. To see this, fix  $v_1$  and  $v_n$ , the remaining node combinations in the middle will be the link between pancake  $P_{n-1}|v_n$  and  $P_{n-1}|v_1$ . There are  $n - 1$  such sets in  $P_{n-1}|v_n$ , each connecting with another  $(n - 1)$ -dimensional pancake.

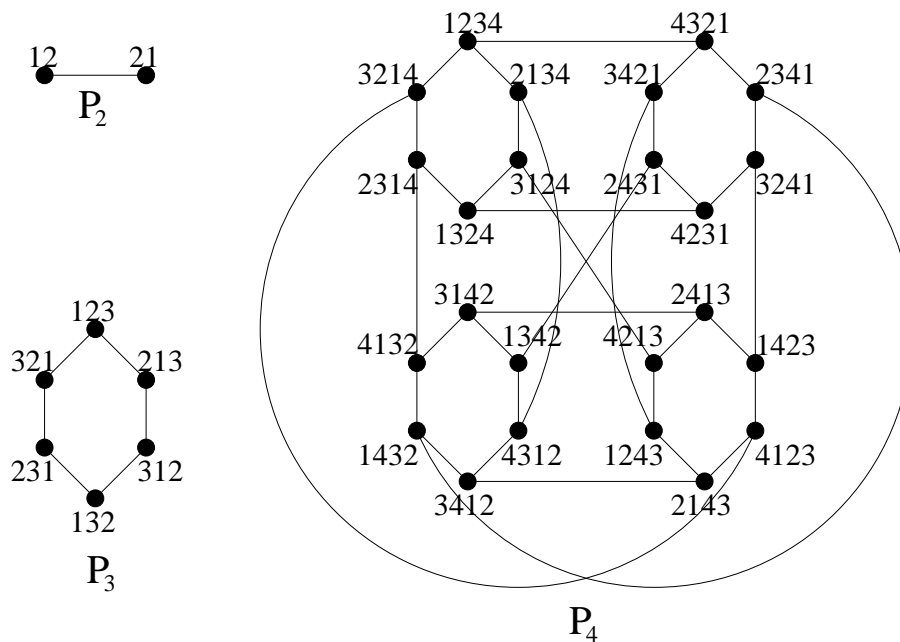


Figure 1: Pancake graphs for  $n = 2, 3, 4$ .

- b) Let us look at the second, more intuitive definition (Eq. (3)). Basically, it states that for every node, there exists exactly one edge for every distinct prefix reversal. So the node degree of  $P_n$  can be stated as follows: how many non-trivial prefix reversals are there for a sequence of  $n$  nodes? Answer:  $n - 1$  with edges  $e_2, \dots, e_n$ . Succinctly,

$$\text{deg}(v) = n - 1 \quad \forall v \in V(P_n).$$

Thus the degree of an  $N$ -node pancake graph is in  $O(\log N / \log \log N)$ .

- c) To give an upper bound on the diameter, we need to determine in how many steps, at most, we can go from one node to any other node. Say we want to get from node  $v = v_1 v_2 \dots v_n$  to node  $w = w_1 w_2 \dots w_n$ . As with all hypercube-like graphs, we will proceed by correcting one “coordinate” at a time. In this case, we start at the back. Since the nodes are all permutations, there will exist a  $v_j$  such that  $v_j = w_n$ . Now take the edges  $v \rightarrow e_j \rightarrow e_n$  to get to node  $v^{(1)} = v_n \dots v_{j+1} v_1 v_2 \dots v_{j-1} w_n$ . We can relabel the indices of  $v^{(1)}$  to go again from 1 to  $n-1$ , leave  $w_n$  fixed, find the index  $j$  with  $v_j = w_{n-1}$ , and take the edges  $v^{(1)} \rightarrow e_j \rightarrow e_{n-1}$ . Thus, by induction, we need at most 2 edges per correct target index, and we are done after  $n-1$  steps. Therefore,

$$D(P_n) \leq 2(n-1)$$

that is, the diameter of  $P_n$  is in  $O(\log N / \log \log N)$ .

Gates and Papadimitriou [1] have also shown that this is asymptotically optimal, that is,

$$D(P_n) \geq n.$$

- d) To show that  $P_n$  is Hamiltonian, we proceed by induction on  $n$ . We will actually show the following stronger claim: In  $P_n$ , there exists a Hamiltonian path from  $12 \dots (n-1)n$  to  $n(n-1) \dots 21$  and the cycle is completed by using edge  $e_n$ . Observe that since in  $P_n$  the graph looks the same from every vertex, this also holds for any given vertex  $v_1 v_2 \dots v_n$ .

For  $n=3$ : by direct observation, we have the path  $123 \rightarrow 213 \rightarrow 312 \rightarrow 132 \rightarrow 231 \rightarrow 321$  and the final edge  $321 \rightarrow 123$ .

Assume that  $P_{n-1}$  has such a Hamiltonian path  $H_{n-1}$  from  $v_1 v_2 \dots v_{n-1}$  to  $v_{n-1} \dots v_2 v_1$ . Then we can construct a Hamiltonian path in  $P_n$  by concatenating the Hamiltonian paths of the  $n$   $P_{n-1}$  subgraphs as follows:

$$\begin{aligned} a_n &= 12 \dots (n-1)n \rightarrow H_{n-1} \rightarrow (n-1) \dots 21n = b_n \\ &\quad b_n \rightarrow e_n \rightarrow a_{n-1} \\ a_{n-1} &= n12 \dots (n-2)(n-1) \rightarrow H_{n-1} \rightarrow n-2 \dots 1n(n-1) = b_{n-1} \\ &\quad b_{n-1} \rightarrow e_n \rightarrow a_{n-2} \\ &\quad \vdots \\ a_2 &= 3 \dots (n-1)n12 \rightarrow H_{n-1} \rightarrow 1n(n-1) \dots 32 = b_2 \\ &\quad b_2 \rightarrow e_n \rightarrow a_1 \\ a_1 &= 2 \dots (n-1)n1 \rightarrow H_{n-1} \rightarrow n(n-1) \dots 21 = b_1 \end{aligned}$$

and we complete the cycle with the final  $b_1 \rightarrow a_n$  edge. Or, more formally, set

$$\begin{aligned} a_i &= (i+1)(i+2) \dots n1 \dots (i-1)i \\ b_i &= (i-1) \dots 1n \dots (i+2)(i+1)i \end{aligned}$$

using  $n+1=1$  and  $1-1=n$ . Then, since the  $n$ th coordinate is fixed, the Hamiltonian path  $H_{n-1}$  from  $a_i$  to  $b_i$  is completely contained in  $n-1$  dimensions. Its existence is guaranteed by the induction hypothesis. Thus, the Hamiltonian path in  $n$  dimensions is given by

$$a_n \overset{H_{n-1}}{\cdots} b_n \rightarrow a_{n-1} \overset{H_{n-1}}{\cdots} b_{n-1} \rightarrow a_{n-2} \dots a_2 \overset{H_{n-1}}{\cdots} b_2 \rightarrow a_1 \overset{H_{n-1}}{\cdots} b_1 \rightarrow a_n$$

where  $a_n = 12 \dots (n-1)n$  and  $b_1 = n(n-1) \dots 21$  as required in the claim.

## References

- [1] W. H. Gates, C. H. Papadimitriou, Bounds for sorting by prefix reversal, *Discrete Math.* 27, (1979), 47–57.