1 Flow labeling schemes

Question 1 Check that $R_k$ is reflexive, symmetric and transitive.

- reflexive: $\text{flow}(x, x) = \infty$
- symmetric: the graph is undirected, $\text{flow}(x, y) = \text{flow}(y, x)$
- transitive: consider a path $p = (v_1, v_2, \ldots, v_m)$ from $x$ to $y$ in which $v_1 = x$ and $v_m = y$ and a path $p' = (v'_1, v'_2, \ldots, v'_{m'})$ from $y$ to $z$ in which $v'_1 = y$ and $v'_{m'} = z$. Let $i$ be the largest subscript in $p'$ such that $v'_i \in p$. It is easy to check there is a path $x - v'_i - z$ where $x - v'_i$ is a part of $p$ and $v'_i - z$ is a part of $p'$.

$C_{k+1}$ is a refinement of $C_k$.

Question 2 Add the depth of each vertex into the label, and the depth of the tree is smaller than $m$.

See that

$$\text{flow}_G(v, w) = \text{SepLevel}_T(t(v), t(w)) + 1. \tag{1}$$

The depth of $T_G$ cannot exceed $n\tilde{\omega}$ and every level at most has $n$ nodes, hence the total number of nodes in $T_G$ is $O(n^2\tilde{\omega})$.

Question 3 Cancel all nodes of degree 2 in $T_G$, and add appropriate edge weights ($\tilde{T}_G$).

Now, define $\text{SepLevel}_T(x, y)$ as the weighted depth of $z = \text{lca}(x, y)$, i.e., its weighted distance from the root. We can also extend $\text{SepLevel}$ labeling schemes to weighted trees. For $\tilde{n}$-node trees with maximum weight $\tilde{\omega}$, the labeling size is $O(\log \tilde{n} \log \tilde{\omega} + \log 2 \tilde{n})$.

It is also easy to verify that for two nodes $x, y$ in $G$, the separation level of the leaves $t(x)$ and $t(y)$ associated with $x$ and $y$ in the tree $\tilde{T}_G$ is still related to the flow between the two vertices, similar to Eq. (1).

Finally, note that as $\tilde{T}_G$ has exactly $n$ leaves, and every non-leaf node in it has at least two children, the total number of nodes in $\tilde{T}_G$ is $\tilde{n} \leq 2n - 1$. Moreover, the maximum edge weight in $\tilde{T}_G$ is $\tilde{\omega} \leq n\tilde{\omega}$.

For more details, please check the paper [1] (Section 2).

References