Principles of Distributed Computing  
Exercise 13

1 Flow labeling schemes

In this exercise, we focus on flow labeling schemes. Let $G = \langle V, E, w \rangle$ be a weighted undirected graph where, for every edge $e \in E$, the weight $w(e)$ is integral and represents the capacity of the edge. For two vertices $u, v \in V$, the maximum flow possible between them (in either direction), denoted $\text{flow}(u, v)$, can be defined in this context as follows. Denote by $G'$ the multigraph obtained by replacing each edge $e$ in $G$ with $w(e)$ parallel edges of capacity 1. A set of paths $P$ in $G'$ is edge-disjoint if each edge (with capacity 1) appears in no more than one path $p \in P$. Let $P_{u,v}$ be the collection of all sets $P$ of edge-disjoint paths in $G'$ between $u$ and $v$. Then $\text{flow}(u, v) = \max_{P \in P_{u,v}} \{|P|\}$.

We consider the family $G(n, \hat{\omega})$ of undirected capacitated connected $n$-vertex graphs with maximum (integral) capacity $\hat{\omega}$, and will find flow labeling schemes for this family. Given a graph $G = \langle V, E, w \rangle$ in this family and an integer $1 \leq k$, let us define the following relation:

$$R_k = \{(x, y) | x, y \in V, \text{flow}(x, y) \geq k\}.$$

**Question 1** Show that for every $k \geq 1$, the relation $R_k$ induces a collection of equivalence classes on $V$, $C_k = \{C^1_k, \ldots, C^m_k\}$, such that $C^i_k \cap C^j_k = \emptyset$ (if $i \neq j$) and $\bigcup_i C^i_k = V$. What is the relationship between $C_k$ and $C_{k+1}$?

According to the solution of Question 1, given $G$, one can construct a tree $T_G$ corresponding to its equivalence relations. The $k$’th level of $T$ corresponds to the relation $R_k$. The tree is truncated at a node once the equivalence class associated with it is a singleton. For every vertex $v \in V$, denote by $t(v)$ the leaf in $T_G$ associated with the singleton set $\{v\}$.

For two nodes $x, y$ in a tree $T$ rooted at $r$, we define the separation level of $x$ and $y$, denoted $\text{SepLevel}_T(x, y)$, as the depth of $z = lca(x, y)$, the least common ancestor of $x$ and $y$. I.e., $\text{SepLevel}_T(x, y) = \text{dist}_T(z, r)$, the distance of $z$ from the root.

**Question 2** Show that if there exists a labeling scheme for distance in trees with labeling size $L(\text{dist}, T)$, then there is a labeling scheme for separation level with labeling size $L(\text{SepLevel}, T) \leq L(\text{dist}, T) + \lceil \log m \rceil$ where $m$ is the number of nodes in the tree. Based on this result and Theorem 13.8 (there is an $O(\log^2 n)$ labeling scheme for distance in trees), show that $L(\text{flow}, G(n, \hat{\omega})) = O(\log^2(n\hat{\omega}))$.

**Question 3** Find a more careful design of the tree $T_G$ which can improve the bound on the label size to $L(\text{flow}, G(n, \hat{\omega})) = O(\log n \log \hat{\omega} + \log^2 n)$. Hint: i) consider all nodes of degree 2 in the tree $T_G$ and weighted trees, ii) naturally extend the notion of separation level to weighted rooted trees.

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1 As a convention, $\text{flow}(x, x) = \infty$. 