Distributed Algorithms

- Message Passing
- Shared Memory
Example: Maximal Independent Set (MIS)

- Given a network with \( n \) nodes, nodes have unique IDs.
- Find a Maximal Independent Set (MIS)
  - a non-extendable set of pair-wise non-adjacent nodes
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![Graph diagram]

10

69

11

17

7
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- Traditional (sequential) computation: The simple greedy algorithm finds MIS (in linear time)
What about a Distributed Algorithm?

• Nodes are agents with unique ID’s that can communicate with neighbors by sending messages. In each synchronous round, every node can send a (different) message to each neighbor.
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  - Each round:
    1. send msgs
    2. rcv msgs
    3. compute
A Simple Distributed Algorithm

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS → join MIS
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What’s the problem with this distributed algorithm?

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2. rcv msgs
3. compute
Example

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS → **join MIS**
Example

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- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS $\rightarrow$ join MIS

![Diagram with nodes and edges]

- What if we have minor changes?

![Diagram with nodes and edges]
Example

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS \(\rightarrow\) join MIS

\[ \begin{array}{cccccccc}
69 & 17 & 11 & 10 & 7 & 4 & 3 & 1 \\
\end{array} \]

- What if we have minor changes?

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Example

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What if we have minor changes?

[Diagram of a network with nodes labeled 69, 17, 11, 10, 7, 4, 3, 1, showing different colors for nodes with different values.]
Example

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS → join MIS

What if we have minor changes?
Example

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What if we have minor changes?
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![Diagram with numbered nodes]

- What if we have minor changes?

![Diagram with numbered nodes and butterfly]

- Proof by animation: In the worst case, the algorithm is slow (linear in the number of nodes). In addition, we have a terrible „butterfly effect“.
What about a **Fast** Distributed Algorithm?

- Can you find a distributed algorithm that is *polylogarithmic* in the number of nodes $n$, for any graph?
What about a **Fast** Distributed Algorithm?

- Surprisingly, for **deterministic** distributed algorithms, this is an open problem!

- However, **randomization** helps! In each synchronous round, nodes should choose a random value. If your value is larger than the value of your neighbors, join MIS!
What about a Fast Distributed Algorithm?

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- How many synchronous rounds does this take in expectation (or whp)?
Analysis

- Event \((u \rightarrow v)\) : node \(u\) got largest random value in combined neighborhood \(N_u \cup N_v\).
- We only count edges of \(v\) as deleted.

- Similarly event \((v \rightarrow u)\) deletes edges of \(u\).
- We only double-counted edges.
- Using linearity of expectation, in expectation at least half of the edges are removed in each round.
- In other words, whp it takes \(O(\log n)\) rounds to compute an MIS.
Results: MIS

General Graphs, Randomized
[Alon, Babai, and Itai, 1986]
[Israeli and Itai, 1986]
[Luby, 1986]
[Métivier et al., 2009]

Decomposition, Determ.
[Awerbuch et al., 1989]
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Naïve Algo
Local Algorithms

- Each node can exchange a message with all neighbors, for \( t \) communication rounds, and must then decide.
- Or: Given a graph, each node must determine its decision as a function of the information available within radius \( t \) of the node.
- Or: Change can only affect nodes up to distance \( t \).
- Or: ...
Locality is Everywhere!

- Self-Assembling Robots
- Applications e.g. Multicore
- Self-Stabilization
- Local Algorithms
- Dynamics
- Sublinear Algorithms
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What about an **Even Faster** Distributed Algorithm?

- Since the 1980s, nobody was able to improve this simple algorithm.

- What about lower bounds?

- There is an interesting lower bound, essentially using a Ramsey theory argument, that proves that an MIS needs at least $\Omega(\log^* n)$ time.
  - $\log^*$ is the so-called iterated logarithm – how often you need to take the logarithm until you end up with a value smaller than 1.
  - This lower bound already works on simple networks such as the linked list.
Coloring Lower Bound on Oriented Ring

- Build graph $G_t$, where nodes are possible views of nodes for distributed algorithms of time $t$. Connect views that could be neighbors in ring.
- Here is for instance of $G_1$:

![Graph Diagram]

- Chromatic number of $G_t$ is exactly minimum possible colors in time $t$. 
Coloring Lower Bound on Oriented Ring

• Build graph $G_t$, where nodes are possible views of nodes for distributed algorithms of time $t$. Connect views that could be neighbors in ring.
• Here is for instance of $G_1$:

```
1  2  3
```

```
2  3  6
```

```
3  6  7
```

```
3  6  9
```

• Chromatic number of $G_t$ is exactly minimum possible colors in time $t$. 
Results: MIS

1. **Log* n**
   - General Graphs, Randomized
     - [Alon, Babai, and Itai, 1986]
     - [Israeli and Itai, 1986]
     - [Luby, 1986]
     - [Métivier et al., 2009]

2. **Log n**
   - Naïve Algo

3. **n^ε**
   - Decomposition, Deterministic
     - [Awerbuch et al., 1989]
     - [Panconesi et al., 1996]

4. **n**
   - Linked List
     - [Linial, 1992]
Results: MIS

1. **Linked List, Deterministic** [Cole and Vishkin, 1986]
   - Linked List [Linial, 1992]

2. **General Graphs, Randomized** [Alon, Babai, and Itai, 1986]
   - [Israeli and Itai, 1986]
   - [Luby, 1986]
   - [Métivier et al., 2009]

3. **Decomposition, Determined** [Awerbuch et al., 1989]
   - [Panconesi et al., 1996]

4. **Naïve Algo**
Results: MIS

\[ |IS(N_2)| \in O(1) \]

- Growth-Bounded Graphs [Schneider et al., 2008]
- Linked List, Deterministic [Cole and Vishkin, 1986]

- General Graphs, Randomized
  - [Alon, Babai, and Itai, 1986]
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- Naïve Algo

- Linked List [Linial, 1992]
Results: MIS

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- **General Graphs, Randomized** [Alon, Babai, and Itai, 1986] [Israeli and Itai, 1986] [Luby, 1986] [Métivier et al., 2009]
- **Other problems** e.g., [Kuhn et al., 2006]
- **Decomposition, Deterministic** [Awerbuch et al., 1989] [Panconesi et al., 1996]
- **Naïve Algo**

- **Linked List** [Linial, 1992]
- **E.g., coloring, CDS, matching, max-min LPs, facility location**
- **E.g., covering/packing LPs with only local constraints: constant approximation in time $O(\log n)$ or $O(\log^2 \Delta)$**

Time Complexity:
- $1, \log^* n, \log n, n^\epsilon, n$
Results: MIS

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Naïve Algo

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**Timeline**

1. Linked List [Linial, 1992]
2. General Graphs [Kuhn et al., 2004, 2006]
3. Naïve Algo
Example: Minimum Vertex Cover (MVC)

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- Find a **Minimum Vertex Cover (MVC)**
  - a minimum set of nodes such that all edges are adjacent to node in MVC
Differences between MIS and MVC

- Central (non-local) algorithms: MIS is trivial, whereas MVC is NP-hard.
- Instead: Find an MVC that is “close” to minimum (approximation).
- Trade-off between time complexity and approximation ratio.

- MVC: Various simple (non-distributed) 2-approximations exist!
- What about distributed algorithms?!!
Finding the MVC (by Distributed Algorithm)

- Given the following bipartite graph with $|S_0| = \delta |S_1|
- The MVC is just all the nodes in $S_1$
- Distributed Algorithm...
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- Distributed Algorithm...
$N_2(\text{node in } S_0)$

$N_2(\text{node in } S_1)$
Graph is “symmetric”, yet highly non-regular!
Lower Bound: The Argument

- The example graph is for $t = 3$.
- All edges are in fact special bipartite graphs with large enough girth.

- If you use the graph of recursion level $t$, then a distributed algorithm cannot find a good MVC approximation in time $t$. 
Lower Bound: The Math

- Choose degrees $\delta_i$ such that $\delta_{i+1}/\delta_i = 2^i \delta$.
- We have $|S_0| > \delta/2 \ |L_1|$, with $|L_1|$ nodes on level 1.
Lower Bound: The Math

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By induction we have a $(1 - \Theta(1/\delta))$ fraction of the nodes is in $S_0$.
- Now $\delta, n, \Delta$ are depending on the recursion level $t$. 
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- Now $\delta, n, \Delta$ are depending on the recursion level $t$. 
Lower Bound: Results

- We can show that for \( \epsilon > 0 \), in \( t \) time, the approximation ratio is at least

\[
\Omega \left( n^{\frac{1}{4} - \frac{\epsilon}{t^2}} \right) \quad \text{and} \quad \Omega \left( \Delta^{\frac{1 - \epsilon}{t + 1}} \right)
\]

- Constant approximation needs at least \( \Omega(\log \Delta) \) and \( \Omega(\sqrt{\log n}) \) time.
- Polylog approximation \( \Omega(\log \Delta / \log \log \Delta) \) and \( \Omega(\sqrt{\log n / \log \log n}) \).
Lower Bound: Results

• We can show that for $\varepsilon > 0$, in $t$ time, the approximation ratio is at least

$$\Omega \left( n^{\frac{1}{4}-\varepsilon} \frac{1}{t^2} \right) \text{ and } \Omega \left( \Delta^{\frac{1}{t+1}} \right)$$

tight for MVC

• Constant approximation needs at least $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time.

• Polylog approximation $\Omega(\log \Delta / \log \log \Delta)$ and $\Omega(\sqrt{\log n / \log \log n})$. 
Many "local looking" problems need non-trivial $t$, in other words, the bounds $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ hold for a variety of classic problems.
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Naïve Algo

\[ \frac{\sqrt{\log n}}{\log n} \ldots \log n \]

Linked List [Linial, 1992]

Open

General Graphs [Kuhn et al., 2004, 2006]
Summary

1. $\log^* n$

- Growth-Bounded Graphs (various problems)
  - E.g., dominating set approximation in planar graphs

- Approximations of dominating set, vertex cover, etc.

- Covering and packing LPs
  - E.g., dominating set approximation in planar graphs

- $\sqrt{\log n} \ldots \log n$

- MIS, maximal matching, etc.

- MST, Sum, etc.

- Diameter
Thank You!
Questions & Comments?

Thanks to my co-authors
Fabian Kuhn
Thomas Moscibroda
Johannes Schneider

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Open Problems

• Close the gap between $\sqrt{\log n}$ and $\log n$ (for randomized algorithms)!
• Find a fast deterministic MIS algorithm (or strong det. lower bound)!
• Where are the boundaries between constant, $\text{log}^*$, $\log$, and diameter?
• What about algorithms that cannot even exchange messages?
• Can the lower bound graph be used in the context of sublinear algorithms?