CHAPTER 9. SOCIAL NETWORKS

## Small World Networks 9.1

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Guglielmo Marconi's 1909 Nobel Prize speech. Despite physical distance, the tulated that the world was "shrinking" because human beings were connected more and more. Some claim that he was inspired by radio network pioneer growing density of human "networks" renders the actual social distance smaller and smaller. As a result, it is believed that any two individuals can be connected Back in 1929, Frigyes Karinthy published a volume of short stories that posthrough at most five (or so) acquaintances, i.e., within six hops.

neous reaction times of new ("electric") technologies, each individual inevitably feels the consequences of his actions and thus automatically deeply participates real and virtual world are moving together. He realized that the transmission medium, rather than the transmitted information is at the core of change, as The topic was hot in the 1960s. For instance, in 1964, Marshall McLuhan the globe is no more than a village". He argues that due to the almost instantain the global society. McLuhan understood what we now can directly observe – coined the metaphor "Global Village". He wrote: "As electrically contracted. expressed by his famous phrase "the medium is the message".

However, they were not allowed to *directly* send the letter, rather, they had to pass it to somebody they knew on first-name basis and that they thought to until somebody knew the target person, and could deliver the letter. Shortly first by Michael Gurevich, later by Stanley Milgram. Milgram wanted to know periments, generally using randomly chosen individuals from the US Midwest after starting the experiment, letters have been received. Most letters were lost The observation that the entire population is connected by short acquaintance This idea has been followed ardently in the 1960s by several sociologists, the average path length between two "random" humans, by using various ex-The starting points were given name, address, occupation, plus some personal information about the target. They were asked to send a letter to the target. have a higher probability to know the target person. This process was repeated, during the process, but if they arrived, the average path length was about 5.5. chains got later popularized by the terms "six degrees of separation" and "small as starting points, and a stockbroker living in a suburb of Boston as target. world".

Statisticians tried to explain Milgram's experiments, by essentially giving community in statistical physics that tries to understand network properties network models that allowed for short diameters, i.e., each node is connected to each other node by only a few hops. Until today there is a thriving research that allow for "small world" effects.

One of the keywords in this area are power-law graphs, networks were node degrees are distributed according to a power-law distribution, i.e., the number of nodes with degree  $\delta$  is proportional to  $\delta^{-\alpha}$ , for some  $\alpha > 1$ . Such powerlaw graphs have been witnessed in many application areas, apart from social networks also in the web, or in Biology or Physics.

Obviously, two power-law graphs might look and behave completely differently, even if  $\alpha$  and the number of edges is exactly the same.

and Strogatz argued that social networks should be modeled by a combination of One well-known model towards this end is the Watts-Strogatz model. Watts two networks: As the basis we take a network that has a large cluster coefficient

## Chapter 9

# Social Networks

Distributed computing is applicable in various contexts. This lecture exemplarily studies one of these contexts, social networks, an area of study whose origins date back a century. To give you a first impression, consider Figure 9.1.



Figure 9.1: This graph shows the social relations between the members of a karate club, studied by anthropologist Wayne Zachary in the 1970s. Two people (nodes) stand out, the instructor and the administrator of the club, both happen to have many friends among club members. At some point, a dispute caused the club to split into two. Can you predict how the club partitioned? (If not, just search the Internet for Zachary and Karate.)

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9.1. SMALL WORLD NETWORKS

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**Definition 9.1.** The cluster coefficient of a network is defined by the probability that two friends of a node are likely to be friends as well, averaged over all the nodes.

..., then we augment such a graph with random links, every node for instance points to a constant number of other nodes, chosen uniformly at random. This augmentation represents acquaintances that connect nodes to parts of the network that would otherwise be far away.

#### Remarks:

Without further information, knowing the cluster coefficient is of questionable value: Assume we arrange the nodes in a grid. Technically, if we connect each node to its four closest neighbors, the graph has cluster coefficient 0, since there are no triangles; if we instead connect each node with its eight closest neighbors, the cluster coefficient is 3/7. The cluster coefficient is quite different, even though both networks have similar characteristics.

This is interesting, but not enough to really understand what is going on. For Milgram's experiments to work, it is not sufficient to connect the nodes in a certain way. In addition, the nodes *themselves* need to know how to forward a message to one of their neighbors, even though they cannot know whether that neighbor is really closer to the target. In other words, nodes are not just following physical laws, but they make decisions themselves.

Let us consider an artificial network with nodes on a grid topology, plus some additional random links per node. In a quantitative study it was shown that the random links need a specific distance distribution to allow for efficient greedy routing. This distribution marks the sweet spot for any navigable network. **Definition 9.2** (Augmented Grid). We take  $n = m^2$  nodes  $(i, j) \in V = \{1, \ldots, m\}^2$  that are identified with the lattice points on an  $m \times m$  grid. We define the distance between two nodes (i, j) and  $(k, \ell)$  as  $d((i, j), (k, \ell)) = |k - i| + |\ell - j|$  as the distance between them on the  $m \times m$  lattice. The network is modeled using a parameter  $\alpha \ge 0$ . Each node u has a directed edge to every lattice neighbor. These are the local contacts of a node. In addition, each onde also has an additional random link (the long-range contact). For all u and v, the long-range contact of u points to node v with probability proportional to  $d(u, v)^{-\alpha}$ , i.e., with probability  $d(u, v)^{-\alpha}/\sum_{w \in V \setminus \{u\}} d(u, w)^{-\alpha}$ . Figure 9.2 illustrates the model.

#### Remarks:

- The network model has the following geographic interpretation: nodes (individuals) live on a grid and know their neighbors on the grid. Further, each node has some additional acquaintances throughout the network.
- The parameter  $\alpha$  controls how the additional neighbors are distributed across the grid. If  $\alpha = 0$ , long-range contacts are chosen uniformly at random (as in the Watts-Strogatz model). As  $\alpha$  increases, long-range contacts become shorter on average. In the extreme case, if  $\alpha \to \infty$ , all long-range contacts are to immediate neighbors on the grid.





Figure 9.2: Augmented grid with m = 6

• It can be shown that as long as  $\alpha \leq 2$ , the diameter of the resulting graph is polylogarithmic in n (polynomial in  $\log n$ ) with high probability. In particular, if the long-range contacts are chosen uniformly at random  $(\alpha = 0)$ , the diameter is  $\mathcal{O}(\log n)$ .

Since the augmented grid contains random links, we do not know anything for sure about how the random links are distributed. In theory, all links could point to the same node! However, this is almost certainly not the case. Formally this is captured by the term with high probability. **Definition 9.3** (With High Probability). Some probabilistic event is said to occur with high probability (w.h.p.), if it happens with a probability  $p \ge 1 - 1/n^c$ , where c is a constant. The constant c may be chosen arbitrarily, but it is considered constant with respect to Big-O notation.

#### Remarks:

- For instance, a running time bound of  $c \log n$  or  $e^{cl} \log n + 5000c$  with probability at least  $1 1/n^c$  would be  $O(\log n)$  w.h.p., but a running time of  $n^c$  would not be O(n) w.h.p. since c might also be 50.
- This definition is very powerful, as any polynomial (in n) number of statements that hold w.h.p. also holds w.h.p. at the same time, regardless of any dependencies between random variables!

Lemma 9.6. Node u's random link points to a node v with pr	• $\Theta(1/(d(u, v)^{\alpha}m^{2-\alpha}))$ if $\alpha < 2$ .	• $\Theta(1/(d(u, v)^2 \log n))$ if $\alpha = 2$ ,	f $\bullet \Theta(1/d(u,v)^{\alpha}) \ if \ \alpha > 2.$	Moreover, if $\alpha > 2$ , the probability to see a link of length at $\Theta(1/d^{\alpha-2})$ .	I Proof. For a constant $\alpha \neq 2$ , we have that	$\sum_{w \in V(1,u)} \frac{1}{d(u,w)^{\alpha}} \in \sum_{r=1}^{m} \frac{\Theta(r)}{r^{\alpha}} = \Theta\left(\int_{r=1}^{m} \frac{1}{r^{\alpha-1}} dr\right) = \Theta\left(\left[\frac{r}{2}\right]$	If $\alpha < 2$ , this gives $\Theta(m^{2-\alpha})$ , if $\alpha > 2$ , it is in $\Theta(1)$ . If $\alpha = 2$ , we	$\sum_{\substack{n \in \mathbb{Z} \\ a \in \mathbb{Z}}} \frac{1}{d(u,w)^{a}} \in \sum_{\substack{n \in \mathbb{Z} \\ x^{2}}} \frac{\Theta(r)}{r} = \Theta(1) \cdot \sum_{n}^{m} \frac{1}{r} = \Theta(\log m) = r$		Multiplying with $d(u, v)^{\alpha}$ yields the first three bounds. For the compute	$\sum_{v \in V} \Theta(1/d(u, v)^{\alpha}) = \Theta\left(\int_{r=d}^{m} \frac{r}{r^{\alpha}} dr\right) = \Theta\left(\left[\frac{r^{2-\alpha}}{2-\alpha}\right]_{d}^{m}\right) = \Theta\left(\left[\frac{r^{2-\alpha}}{2-\alpha}\right$	$b \leq (u,v) \geq d$	Remarks:	• If $\alpha > 2$ , according to the lemma, the probability to see of length at least $d = m^{1/(\alpha-1)}$ is $\Theta(1/m^{\alpha-2}) = \Theta(1/m^{1/2})$	expectation we have to take $\Theta(m^{(\alpha-2)}/(\alpha^{-1}))$ hops until we link of length at least $d$ . When just following links of leng it takes more than $m/d = m/m^{1/(\alpha-1)} = m^{(\alpha-2)/(\alpha-1)}$ hoords, in expectation, either way we need at least $m^{(\alpha-2)/(\alpha-1)}$ hops to the destination.	• If $\alpha < 2$ , there is a (slightly more complicated) argument.	a border around the nodes in distance $m^{(2-\alpha)/3}$ to the targe
the intermentate channs notes with mgn probability, which then by means union bound yields that all of the claims hold simultaneously with high	bility for all pairs of nodes (see exercises). $t = M$ be the $\lceil \log n \rceil$ -horn neichborhood of source $s$ on the order containing	$^{2}n_{s}$ be the $\log n$ -hole negation node of some s on the grid, containing $^{2}n$ ) nodes. Each of the nodes in $N_{s}$ has a random link, probably leading	tant parts of the graph. As long as we have reached only $o(n)$ nodes, any andom link will with probability $1 - o(1)$ lead to a node for which none of	d neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ odds whose neighbors are "fresh". Using their grid links, we will reach	$(1)$ $ N_{\rm s} $ more nodes within one more hop. If bad lick strikes, it could still in that many of these links lead to a few nodes, already visited nodes, or	that are very close to each other. But that is very minkery, as we have frandom choices! Indeed, it can be shown that not only in expectation, ith high probability $(5 - o(1)) N_s $ many nodes are reached this way (see	ses). cause all the new nodes have (so far unused) random links, we can repeat	asoning inductively, implying that the number of nodes grows by (at least) astant factor for every two hops. Thus, after $O(\log n)$ hops, we will have $\operatorname{ed} n/\log n$ nodes (which is still small compared to $n$ ). Finally, consider the	ted number of links from these nodes that enter the $(\log n)$ -neighborhood to target node t with respect to the original Since this neighborhood consists	$\alpha^2$ on some restriction of the production of	is to range enough to almost guarance that the improve two vertices), ing everything up, we still used merely $O(\log n)$ hops in total to get from		is shows that for $\alpha = 0$ (and in fact for all $\alpha \leq 2$ ), the resulting network small diameter. Recall however that we also wanted the network to be ble. For this, we consider a simple greedy routing strategy (Algorithm 39).	ithm 39 Greedy Routing	ille not at destination do go to a neighbor which is closest to destination (considering grid distance only) d while	0.6. To the accompany of anid Alexantifier 30 fords a montion wath of loweth	to $2.5.$ In the augmented grave, regorements of forms a rounning punit of tength $t \ 2(m-1) \in O(\sqrt{n}).$
<b>rem 9.4.</b> The diameter of the augmented grid with $\alpha = 0$ is $O(\log n)$ with $or = 0$ is $O(\log n)$ . This is that the random link is the process. For $O(\log n)$ we will only show that we can reach a target node integration of $O(\log n)$ in terms of their grid distance $d(u, v)$ , the number the form some source node $S$ . However, it can be shown that (estimately or $O(\log n)$ in the constant parameter $\alpha$ .	<b>rem 9.4.</b> The diameter of the augmented grid with $\alpha = 0$ is $O(\log n)$ with orobability. This is not really what Milgram's experiment promises. 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The dimeter of the expense of set of the set of the softificant is realized in the form in the process. The second make speed up the process of the set of understand low law is readom links speed up the process. The second make speed make set is a match make speed up the process of the set of the softificant is another in the softificant is a soft of the species. The softificant is a soft of the soft is the soft of	<b>u</b> 0.4. <i>The functor of the contrant products well only show that we can reach a target noise when the propertional products of the propertional products of the production products of the production product product products of the product produ</i>	<b>u 6.4</b> <i>The standard yeld wath a constraint production of the constraint additional readom links speed up the process. We must a start a matter at the readom link of speed up the standard production in the constraint protoner constraints and standard production constraints and standard production in the constraint protoner constraints and standard production constraints and standard production constraints and standard production constraints are constraints and standard production constraints are constraint. The constraints are constraint protoner constraints and standard production constraints are constraints and the constraints are constraints and and the constraints are constraints and and the constraints are constraints a</i>	$ \begin{array}{llllllllllllllllllllllllllllllllllll$	$ \begin{array}{llllllllllllllllllllllllllllllllllll$
Sketch. For simplicity, we will only show that we can reach a target node the random multiplicity, we will only show that we can reach a target node the random multiplicity we will only show that (essentially) the number that from some source node $v$ , in terms of their grid distance $d(u, v)$ , the number that from some source node $s$ . However, it can be shown that (essentially) the number of $(v, v)$ the nu	<i>Sketch.</i> For simplicity, we will only show that we can reach a target node the random must be an end of $(u, v)$ , the numbe ting from some source node $v$ , in terms of their grid distance $d(u, v)$ , the numbe ting from some source node $s$ . However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means $v$ union bound yields that <i>all</i> of the claims hold simultaneously with high	Sketch. For simplicity, we will only show that we can reach a target node the fraction of the reaction that we can reach a target node the reaction that (essentially) the number ting from some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability which then by means the intermediate claims holds with high probability which then by means the intermediate claims holds with high probability of the claims hold simultaneously with high high billy for a node with $1 + 0$ . Node u's random link points to a node with h billy for all points the area of the claims for a node with h high billy for a node with high billy for a node with high billy for the claims hold simultaneous in the need of the claims hold simultaneous in the need of the claims hold simultaneous the need of the claims hold with hold with hold the claims hold simultaneous the need of the claims hold simultaneous the need of the claims hold simultaneous the need of the claims hold with hold the claims the need of the	<i>Sketch.</i> For simplicity, we will only show that we can reach a target node ting from some source not uncerstant now mery to so that reaction tunce the number ting from some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means to mode with a subsection of the constant parameter $\alpha$ . Intermediate claims holds with high probability, which then by means to mode with a subsection of the constant parameter $\alpha$ . Intermediate claims holds with high probability, which then by means to mode with $i_{i}$ and $i_{i}$ standard the claims hold simultaneously with high billy for all pairs of nodes (see exercises). In the pairs of nodes (see exercises). In nodes. Each of the nodes in $N_s$ has a random link, probably leading $2^n$ ) nodes. Each of the nodes in $N_s$ has a random link, probably leading $-2^n$ , nodes. Each of the nodes in $N_s$ has a random link, probably leading $-2^n$ .	Sketch. For simplicity, we will only show that we can reach a target node ting from some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means to mode v in terms of their grid distance $d(u, v)$ , the numbe the constant parameter $\alpha$ . Intermediate claims hold simultaneously with high bility for all pairs of nodes (see exercises). Intermediate claims hold simultaneously with high bility for all pairs of nodes (see exercises). Intermediate claims hold simultaneously with high bility for all pairs of nodes (see exercises). Intermediate claims hold simultaneously with high bility for all pairs of nodes (see exercises). Intermediate claims hold simultaneously with high is $N_s$ be the [log $n$ ]-hop neighborhood of source $s$ on the grid, containing $t N_s$ be the [log $n$ ]-hop neighborhood of source $s$ on the grid, containing $t N_s$ be the flog $n$ ]-hop neighborhood of source $s$ on the grid, containing $t N_s$ be the graph. As long as we have reached only $o(n)$ nodes, any that parts of the action in $N_s$ have to a node for which none of andom link will with probability $1 - o(1)$ lead to a node for which none of	<i>Sketch.</i> For simplicity, we will only show that we can reach a target node ting from some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability and the means in the constant parameter $\alpha$ . Intermediate claims holds that all of the claims hold simultaneously with high billy for all pairs of nodes (see exercises). Intermediate claims holds that all of the claims hold simultaneously with high billy for all pairs of nodes (see exercises). Is $N_s$ be the $\lceil \log n \rceil$ -hop neighborhood of source s on the grid, containing the intermediate claims in $N_s$ has a random link, probably leading that all of the nodes in $N_s$ has a random link, probably leading that parts of the graph. As long as we have reached only $\sigma(n)$ nodes, any andom link will with probability $1 - \sigma(1)$ lead to a node for which none of d neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ does whose neighbors are "fresh". Using their grid links, we will reach only of the nodes in $N_s$ for $2$ , the probability to see a link of length of the nodes in $N_s$ for $2$ .	<i>Sketch.</i> For simplicity, we will only show that we can reach a target node ting from some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means the intermediate claims holds with high probability, which then by means to mode with some source node s. However, it can be shown that (essentially) of the intermediate claims holds with high probability, which then by means to mode with parameter $\alpha$ . Introduce the claims hold simultaneously with high billy for <i>all</i> parameter $\alpha$ . Introduce the claims hold simultaneously with high billy for <i>all</i> pairs of nodes (see exercises). Introduce the nodes (see exercises). Intermediate claims holds in <i>N</i> , probably leading the <i>N</i> are andom link, probably leading to <i>N</i> are andom link, probably leading that parts of the graph. As long as we have reached only o( <i>n</i> ) nodes, any andom link will with probability $1 - o(1)$ lead to a node for which more of anothes in <i>N</i> and a most $ N_s $ nodes whose neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ nodes whose neighbors are "fresh". Using their grid links, we will reach only of the nodes, or the graph. As long as a reached node of a wile none of a neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ nodes whose neighbors are "fresh". Using their grid links, we will reach only of these links lead to a four of the nodes, are that and that many of these links lead to a four desting the nodes, or that many of these links lead to a for whose neighbors are "fresh". Using their grid links, the nucles of the nodes, or the that many of these links lead to a for whose visited nodes, or the nucles of the	Sketch. For simplicity, we will only show that we can reach a target node ting from some source node s. However, it can be shown that (essentially) the number the nonder with high probability, which hen by means union bound yields that all of the claims holds with high probability, which hen by means union bound yields that all of the claims hold simultaneously with high probability. All the number the constant parameter $\alpha$ . <b>Lemma 9.6.</b> Node u's random link points to a node v with 1 billity for all pairs of nodes (see exercises). $t N_s$ be the $\lceil \log n \rceil$ -hop neighborhood of source s on the grid, containing $^2n$ nodes. Each of the nodes in $N_s$ has a random link, probably leading taut parts of the graph. As long as we have meaded only $\sigma(n)$ nodes, any adom link will with probability $1 - \alpha(1)   ada   a - 2$ , $\Theta(1/ d(u, v)^{\alpha}   a^{-2}))$ if $\alpha < 2$ . $d$ neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ nodes whose neighbors are "fresh". Using their grid links, we will reach (1)  $ N_s $ more nodes within one more bop. If bad luck strikes, it could still nodes whose neighbors are "fresh". Using their grid links, we will reach (1)  $ N_s $ more nodes within one more bop. If bad luck strikes, it could still nodes whose neighbors are "fresh". Using their grid links, we will reach (1)  $ N_s $ more nodes within one more bop. If bad luck strikes, it could still nodes whose neighbors are "fresh". Using their grid links, we were that that many of these links of these made on the strikes, it could still nodes whose neighbors are "fresh". Using their grid links, we have reached this way (see fraction, $\frac{1}{10^{(n-2)}}$ . The probability $1 - \alpha > 2$ , the probability $1 - \alpha > 2$ , the mombility $1 - \alpha > 2$ , the arbody link $\frac{1}{10^{(n-2)}}$ . The mombility $\frac{1}{10^{(n-2)}}$ if $\frac{1}{n}$ i	Sketch. For simplicity, we will only show that we can reach a target node migrit finance diverse it can be shown that (sesentially) fit is more dealers). The the new source node s. However, it can be shown that (sesentially) fit is more dealers) if the intermediate chains holds with high probability, which then by means union bound yields that all of the claims hold simultaneously with high probability which then by means union bound yields that all of the claims hold simultaneously with high probability which then by means union bound yields that all of the claims hold simultaneously with more holds when the transmiss of mode s (a. with a node with a secretises). <b>Lemma 9.6.</b> Node u's random link points to a node v with 10 holds and the nodes in $N_{10}$ has a random link probaby beading $\tau^{(1)}$ nodes $(m^{(1)}, m^{(2)}) = 0$ . $\Theta(1/d(u, v)^2) = 0$ , $M_{10} = 2$ , $\Theta(1/d(u, v)^2) = 0$ , $M_{10} = 2$ , $\Theta(1/d(u, v)^2) = 0$ , $M_{10} = 2$ , $\Theta(1/d(u, v)^2) = 0$ , $M_{10} = 2$ , $M_{10} =$	Stetch. For simplicity, we will only show that we can reach a target node in form some some mode s. Howver, it can be shown that (securital) the intermediate claims holds with high the intermediate claims holds with high muion bound yields that $all$ of the claims hold simultaneously with high the intermediate claims hold simultaneously with high the intermediate claims hold simultaneously with high which here is a mode with promote a sum the react and the intermediate claims hold simultaneously with high the intermediate claims hold simultaneously with here with a standom link, probably leading that period of source soute grid, contraining that pract of the prodes in $N_{\rm c}$ has a random link, probably leading tauto parts of the graph. Also noted source soute grid, conditating the regulation of source soute grid, conditating the regulation are noted for which more of the node link in the probability $1 - \alpha(1)$ had note strikes, it can be shown that none of these finds of a mode a visited that many of these links lead to a few value that are very cooks. Indeed, it can be shown that no only in expectation, the high probability $(5 - \alpha(1)) N_{\rm s} $ many nodes are reached this way (see see). If $\alpha < 2$ , this gives $\Theta(m^{2-\alpha})$ , if $\alpha > 2$ , the probability $\alpha > 2$ , escaled the number of nodes grows by (at least) second in that many of these finds franks in $\Theta(1)$ , $\frac{1}{m^{2}}$ , $\frac$	Stetch. For simplicity, we will only show that we can reach a target node account and simulations of the nodes in them so that we will not here by a semialty in the intermediate calls in them so that we can reach a target node account and simultaneously with high points in them so that them by means a union bound yields that all of the claims hold simultaneously with high more nodes. However, it can be shown that (see servises). Billy for all physic of nodes (see cercises). Billy for all physic of nodes (see cercises). Two here the log null physic nodes (see cercises). Two houses Each of the nodes in <i>N<sub>c</sub></i> has a random link, probably leading that all of the nodes in <i>N<sub>c</sub></i> has a random link, probably leading that all of the nodes in <i>N<sub>c</sub></i> has a random link probably leading that all of the nodes in <i>N<sub>c</sub></i> has a random link the nodes in <i>N<sub>c</sub></i> has a r	Starts. For simplicity, we will only show that we can reach a target node in terms of their grid distance (d, v), the number the impression model with high probability which then by means the feast milding the feast model with high probability which then by means the improvementates can not extra constant parameter $\alpha$ . If the intermediate claims holds with high probability which then by means in the constant parameter $\alpha$ . If the intermediate claims holds with high probability which then by means in the points to a node $u$ with $1$ pairs of nodes $i_{1}$ . The mean of the regulation of sources and the grid volution note of nodes, any maximum to node $u$ with $1$ points for a node $u$ with $1$ probability for $2$ points for a node $u$ with $1$ probability for $1$ pairs of nodes $i_{1}$ . The mean of the regulation of sources and the grid volution note of nodes, $u_{1}$ that makes $u_{1}$ and pairs of nodes $u_{2}$ of $u_{1}$ and $u_{2}$ and $u_{1}$ and $u_{2}$ and $u_{2}$ and $u_{1}$ and $u_{2}$ and $u_{2}$ and $u_{1}$ and $u_{2}$ a	Setch. For simplicity, we will only show that we can reach a target note the formate source source many transmission. The intermediate claims holds with high probability, which then by means a more than the off of the moments and the oright of the main the oright of the main the points to a node w with main the intermediate claims holds with high probability, which then by means a more poster claims holds with high probability which then the intermediate claims holds with high probability has a random link, probabily heading and prate it for the ordes in the probability is the automation in with the probability is the main and automation in with the direct and at most [X]. (1) a direct of the nodes with and at most [X] and (1) a direct of the nodes with and and prate the massed head with a more direct of the nodes in the arrited at the arrited at the arrited at the arrited at the arrited at the arrited at the arrited at the arrited at the arrited at the arrited at the arread at the arrited at the a	Satch. For simplicity, we will only show that we can reach a target node matrix from some some nodes. Numerex, it can be shown that free standary) the intermediate calculates and indiventing high probability, which then by measures or in the intermediate calculates and indiventing the probability with the probability of the number of nodes. Each of the edge $n_1$ we can reach a target node in your potentiate calculates and $n_1$ with the probability with the probability of $n_2$ . We the log $n_1$ -hop meighborhous of some services $n_2$ are the edge $n_1$ in terms of the edge $n_1$ in terms of the edge $n_1$ in the moment of the edge $n_1$ is the mode $n_1$ in the moment of the edge $n_1$ is the mode $n_1$ in the mode $n_1$ in the moment $n_1$ is probably beading and parts of the edge $n_1$ . Also as we have reached $n_1$ or ( $n_1$ ) holds and $n_1$ is the moment point in the probability of $n_2$ is the indiverse of $n_1$ is the moment $n_1$ is the mode $n_1$ in the moment $n_1$ is the moment $n_1$ is the moment $n_1$ is the mode $n_1$ in terms of the edge $n_1$ is $n_2 = 2$ . Indicating the edge $n_1$ is the moment $n_1$ is the number of $n_2$ is the probability to see a link of length $n_1$ is the are very close a dired $n_1$ is $n_2$ is $n_1$ is $n_2 = 2$ . In that we very close $n_1$ is the number of nodes $n_1$ is the number of nodes $n_1$ is the number of $n_2$ is the probability to see a link of length $n_1$ is the number of nodes $n_1$ is the number of nodes $n_2$ is the number of nodes $n_1$ is the number of nodes $n_2$ is the number of number of nodes $n_1$ is the number of nodes $n_1$ is the number of number o	Setch. For simplicity, we valid only short that we can each a target node to the fractional matrix points to a random intro mode with high probability, which then by measures the structure of	Sector for single year of the regard index or the regard index of the regard index or of t	The first project of the state	Back from the strength of work in a constant part of the constant part
	or the intermetate claims notes with much protonously writen used by means is union bound yields that all of the claims hold simultaneously with high	Lemma 9.6. Node $u$ 's random link points to a node $v$ with high bility for all pairs of nodes (see exercises). • $O(1/(d(u, v)^{\alpha}m^{2-\alpha}))$ if $\alpha < 2$ .	Lemma 9.6. Node u's random link points to a node v with picture in the means in the mean point and the claims hold simultaneously with high bility for all pairs of nodes (see exercise). * $N_s$ be the [log n]-hop neighborhood of source s on the grid, containing $^2$ n) nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ n) nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ n) the nodes in $N_s$ has a random link, probably leading $^2$ n) nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ n) nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ n) nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ nodes the log nodes in $N_s$ has a random link, probably leading $^2$ nodes. Each of the nodes in $N_s$ has a random link, probably leading $^2$ nodes.	Lemma 9.6. Node $u's$ random link points to a node $v$ with $p_1$ union bound yields that all of the claims hold simultaneously with high bility for all pairs of nodes (see exercises). $v N_s$ be the [log $n$ ]-hop neighborhood of succes on the grid, containing $^2n$ nodes. Each of the nodes is $n_s$ has a random link, probably leading that not the nodes in $N_s$ has a random link, probably leading that not the nodes is where reached only $o(n)$ nodes, any endom link probability $1 - o(1)$ lead to a node for which none of and mode link will with probability $1 - o(1)$ lead to a node for which none of	Lemma 9.6. Node u's random link points to a node v with probability for all pair-lop neighborhood of source s on the grid, containing $^{2}n$ ) in the claims hold simultaneously with high billity for all pair-lop neighborhood of source s on the grid, containing $^{2}n$ ) nodes. Each of the nodes in $N_s$ has a random link, probably leading tant parts of the graph. As long as we have reached only $o(n)$ nodes, any andom link will with probability $1 - o(1)$ lead to a node for which none of d neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ holds. When we will reach not be noted for which none of d neighbors are "fresh". Using their grid links, we will reach not be noted by the graph. Thus, in expectation we find almost $ N_s $ holds and not be noted by the graph. As long ther graph holds, any andom link will with probability $1 - o(1)$ lead to a node for which none of d neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ holds are always as the fresh. Using their grid links, we will reach not be noted by the graph. Thus, in expectation we find almost $ N_s $ holds are always as a standard by the graph. Thus, in expectation we find almost $ N_s $ holds are always and no be not be n	Lemma 9.6. Node u's random link points to a node v with picture uncentrate charactering into bound yields that all of the claims hold simultaneously with high bility for all pairs of nodes (see exercises). at $N_s$ be the $\lceil \log n \rceil$ -hop neighborhood of source s on the grid, containing $2^n$ ) nodes. Each of the nodes in $N_s$ has a random link, probably leading that pairs of nodes (see exercises). at material random set is exercises). at $N_s$ be the $\lceil \log n \rceil$ -hop neighborhood of source s on the grid, containing $2^n$ ) nodes. Each of the nodes in $N_s$ has a random link, probably leading that pairs of the graph. As long as we have reached only $o(n)$ nodes, any and pairs of the graph. As long as we have reached noted in $o(n)$ nodes, any and the will with will with will with $N_s$ in expectation we find almost $ N_s $ nodes the nodes in $N_s$ has a random link, we will reach of the nodes in $N_s$ is the probability to see a link of length $\Theta(1/d(u, v)^{\alpha})$ if $\alpha > 2$ . Moreover, if $\alpha > 2$ , the probability to see a link of length $\Theta(1/d^{\alpha-2})$ . In that many of these links lead to a few nodes, arrandom fail material $N_s$ in that many of these links lead to a few nodes, or a node so the grid links, we will reach nodes or $(1) N_s $ more nodes within one more hop. If bad luck strikes, it could still and the holes or $(1) N_s $ more nodes within one more hop. If bad nodes, or $(1) N_s $ more nodes within one more hop a few that the holes. Thus the holes of the ender one hop a few that the holes of the enders of the ender of these links lead to a few nodes. The probability to see a link of length and the holes of the enders of the ender of a few the ender of a nodes. If $N_s N_s $ more nodes within one more hop. If bad nodes, or $N_s N_s $ more nodes within one more hop. If how holes, are a nodes of $N_s N_s $ how holes that the thermany of these links lead to a few nodes. If $N_s N_s $ how holes that the hole of $N_s N_s $ how holes that the hole of $N_s N_s $ holes thole of a few nodes. If $N_s N_s $ hole of	Lemma 9.6. Node u's random link points to a node v with protonousy, which used the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity for all pairs of the claims hold simultaneously with high billity is exercises). * $N_s$ be the $[\log n]$ -hop neighborhood of source s on the grid, containing $2^n$ nodes. Each of the nodes in $N_s$ has a random link, probably leading tant parts of the graph. As long as we have reached only $o(n)$ nodes, any andom link will with probability $1 - o(1)$ lead to a node for which none of diverse the form in will with probability $1 - o(1)$ lead to a node for which none of a neighbors has been visited yet. Thus, in expectation we find the almost $ N_s $ are a node in will with probability $1 - o(1)$ lead to a node for which none of a neighbors has been visited yet. Thus, in expectation we for the graph. As long as we have encloses, if $\alpha > 2$ , the probability to see a link of length of $0(1/d^{\alpha-2})$ . Thus, in expectation, if $N_s$ have that are very close to each other. But that is very unlikely, as we have that an every close to each other. But that is very unlikely, as we have for the intervention of the ed, it can be shown that not only in expectation, it thigh probability $(5 - o(1)) N_s $ many nodes are reached this way (see the intervent) is a node of the probability $(5 - o(1)) N_s $ many nodes are reached this way (see the intervent) is the probability $(5 - o(1)) N_s $ many nodes are reached this way (see the length bigh probability $(5 - o(1)) N_s $ many nodes are reached this way (see the length bigh probability $(5 - o(1)) N_s $ many nodes are reached this way (see the length bigh probability $(5 - o(1)) N_s $ many nodes are reached this way (see the len	The memory of the diams holds with many of the claims holds with math and with the detains holds with math disk mutuation of sources on the grid, containing to rall pairs of nodes (see exercise). The Na be the [log r1]-hop neighborhood of source s on the grid, containing to and the visit of the graph. As look are standon link, probably leading tart for all pairs of the graph. As look are standon link, probably leading tart for all pairs of the graph. As look are reached only o(n) nodes, any and non link with probability 1 - o(1) lead to a node for which more of d neighbors has been visited yet. Thus, in expectation we find almost $ N_s $ no node whose neighbors are "fready relied nodes, or does whose neighbors are "fready risked we have that that many of these inits near ordes whose neighbors are fready visited nodes, or that many of the strikes, it could still an that many of these links finds that at a very close to each other. But that is very unlikely, as we have that that many of lead to a shown that not only in expectation, if high probability (5 - o(1)) N_s  more nodes within one loads are reached this way (see section) if $(5 - o(1)) N_s $ many nodes are reached this way (see section). For that are very close to each other. But that is very unlikely, as we have that the nary of hese inks nodes within one loads that any of head loads in the many of head loads in the new nodes and that is very unlikely, as we have that that many of head loads the dual loads transfords to each other. But that is very unlikely, as we have that that are very close to each other. But that is very unlikely, as we have that the term of each other. But that is very unlikely as we have that that are very close to each other. But that is very unlikely, as we hav	Lemma 9.6. Node u's random link points to a node v with prime and of the endes in N <sub>x</sub> has a random link points to a node v with probability for all pairs of nodes (see exercises). 1.N <sub>x</sub> be the [Dg n]-dop registion of source s on the grid, containing to source s on the grid, containing to source s on the grid, containing to node (not not source s) if $\alpha < 2$ . 1.N <sub>x</sub> be the [Dg n]-dop registion of source s on the grid, containing to node for which more of domains, prints of nodes (see exercises). 1.N <sub>x</sub> be the [Dg n]-dop registion of source s on the grid, containing the node for which more of domains, prints 1 = 0.01 lead to a node for which more of domains will with probability 1 = 0.01 lead to a node for which more of domains are "treat". Thus, in expectation we find almost [N <sub>x</sub> ] does use the nodes in N <sub>x</sub> has a random link, points are the node for which more of domains will with probability $1 = 0.01$ lead to a down find almost [N <sub>x</sub> ] does whose registhors are "treat". Thus, in expectation, the ordes, at reach the set is the antis of tength of the enders in the more hop. If had the strikes, it could still undom link will more node whose registhors are "treat". Thus, in expectation, the enders of the enders	The mean state and with the points protonory, with high points and simultaneously with high points bud services). To Notes. Each of the profession N, has a random link probably bedding the product and the modes. The North points point has a random link will with the probability 1 - o(1) bed to a node for which none of dure graph. As long as we have reached only $\sigma(n)$ nodes, any audom link will with probability 1 - o(1) bed to a node for which none of dure graph. As long as we have reached only $\sigma(n)$ nodes, any audom link will with probability 1 - o(1) bed to a node for which none of dure graph. As long as we have reached only $\sigma(n)$ nodes, any audom link will with probability $1 - \sigma(1)$ had not strikes, it could still that the react visited yet. Thus, in expectation, that not only line spress, it could still that a very under the strikes. It could still that a very under the strikes is the norther. But that a very under the strikes is the north still which $\rho(1/q(u, v)^2)$ if $\sigma > 2$ . the probability $1 - \sigma(1) = \rho(1/q(u, v)^2)$ if $\sigma > 2$ . How that not only in expectation, that are very close to each other. But that is very unifiedly, say we have transformed to the strike and the new todes have (set the neotest) and that are very likely as we have the neotest in that are very likely as the neotest in that are very likely for the number of number	The material state of the dams bold with the points to a node v with $p_{1}$ the material poly and the dimet $d_{1}$ is the dimet $d_{2}$ is the table of the dimet $d_{2}$ is the reactive of the graph. As long as we have reached only o(n) nodes, any match with probability $1 - \alpha(1)$ had to a node $t$ with none of malines $1/x_{1}^{1}$ is the dimet $1/x_{1}^{1}$ is the dime	The international yields that all of the claims hold simultacools with high into an order set and with the probability of a contraining in price and pairs of nodes (new corresisa). in price and pairs of nodes (new corresisa). in proceed in the graph. All of a claims hold simultacools with high in proceed in the graph. 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We here [legr]-hop arcgliphorhood (sources on the grid, contains the secretion in the secret se	The intervalue of cancer of the gradient of the data hold stant and the data hold start and hold hold hold hold hold hold hold hol	The contrast contrast and the definition of the second matrix product or a node v with the second matrix product on a mode v with the second matrix product	$ \begin{array}{l} Lemmon of point of constant and product for a notice with product for an one for which means the product for a notice with product for an one for which means the product for a notice with manual of the product for an one for which means the product for a notice material product for an one for which means the product for a notice material product in a notice material product for a notice material pro$

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in expectation, we have to do  $\Theta(m^{(2-\alpha)/3})$  hops. This is too slow, and our greedy strategy is probably faster, as thanks to having  $\alpha < 2$  there are many long-range links. However, it means that we will probably enter the border of the target area on a regular grid link. Once inside the target area, again the probablity of short-cutting our trip by a random long-range link is  $\Theta(1/m^{(2-\alpha)/3})$ , so we probably just follow grid links,  $m^{(2-\alpha)/3} = m^{(2-\alpha)/3}$ .

- In summary, if  $\alpha \neq 2$ , our greedy routing algorithm takes  $m^{\Omega(1)} = n^{\Omega(1)}$  expected hops to reach the destination. This is polynomial in the number of nodes n, and the social network can hardly be called a "small world".
- Maybe we can get a polylogarithmic bound on n if we set  $\alpha=2?$

**Definition 9.7** (Phase). Consider routing from source s to target t and assume that we are at some intermediate node w. We say that we are in phase j at node w if the lattice distance d(w,t) to the target node t is between  $2^{j} < d(w,t) \leq 2^{j+1}$ .

#### Remarks:

- Enumerating the phases in decreasing order is useful, as notation becomes less cumbersome.
- There are  $\lceil \log m \rceil \in O(\log n)$  phases.

**Lemma 9.8.** Assume that we are in phase j at node w when routing from s to t. The probability for getting (at least) to phase j-1 in one step is at least  $\Omega(1/\log n)$ .

*Proof.* Let  $B_j$  be the set of nodes x with  $d(x,t) \leq 2^j$ . We get from phase j to (at least) phase j - 1 if the long-range contact of node w points to some node in  $B_j$ . Note that we always make progress while following the greedy routing path. Therefore, we have not seen node w before and the long-range contact of w points to a random node that is independent of anything seen on the path from s to w.

For all nodes  $x \in B_j$ , we have  $d(w, x) \leq d(w, t) + d(x, t) \leq 2^{j+1} + 2^j < 2^{j+2}$ . Hence, for each node  $x \in B_j$ , the probability that the long-range contact of w points to x is  $\Omega(1/2^{2j+4} \log n)$ . Further, the number of nodes in  $B_j$  is at least  $(2^j)^2/2 = 2^{2j-1}$ . Hence, the probability that some node in  $B_j$  is the long range contact of w is at least

$$\Omega\left(|B_j|\cdot \frac{1}{2^{2j+4}\log n}\right) = \Omega\left(\frac{2^{2j-1}}{2^{2j+4}\log n}\right) = \Omega\left(\frac{1}{\log n}\right).$$

**Theorem 9.9.** Consider the greedy routing path from a node s to a node t on an augmented grid with parameter  $\alpha = 2$ . The expected length of the path is  $O(\log^2 n)$ .

*Proof.* We already observed that the total number of phases is  $\mathcal{O}(\log n)$  (the distance to the target is halved when we go from phase j to phase j - 1). At each point during the routing process, the probability of proceeding to the next phase is at least  $\Omega(1/\log n)$ . Let  $X_j$  be the number of steps in phase j. Because

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the probability for ending the phase is  $\Omega(1/\log n)$  in each step, in expectation we need  $O(\log n)$  steps to proceed to the next phase, i.e.,  $\mathbb{E}[X_j] \in O(\log n)$ . Let  $X = \sum_j X_j$  be the total number of steps of the routing process. By linearity of expectation, we have

$$\mathbb{E}[X] = \sum \mathbb{E}[X_j] \in O(\log^2 n).$$

#### Remarks:

- One can show that the  $\mathcal{O}(\log^2 n)$  result also holds w.h.p.
- In real world social networks, the parameter  $\alpha$  was evaluated experimentally. The assumption is that you are connected to the geographically closest nodes, and then have some random long-range contacts. For Facebook grandpa LiveJournal it was shown that  $\alpha$  is not really 2, but rather around 1.25.

## 9.2 Propagation Studies

In networks, nodes may influence each other's behavior and decisions. There are many applications where nodes influence their neighbors, e.g., they may pase on their opinions, or they may bias what products they buy, or they may pase on a disease. On a beach (modeled as a line segment), it is best to place an ice cream stand right in the middle of the segment, because you will be able to "control" the beach most easily. What about the second stand, where should it settle? The answer generally depends on the model, but assuming that people will buy ice cream from the stand that is closer, it should go right next to the first stand. Burnors can surveise a traveision by fast throuch social networks. The difficu

ice cream from the stand that is closer, it should go right next to the first stand. Rumors can spread surprisingly fast through social networks. Traditionally this happens by word of mouth, but with the emergence of the Internet and its possibilities new ways of rumor propagation are available. People write email, use instant messengers or publish their thoughts in a blog. Many factors influence the dissemination of rumors. It is especially important where in a network a rumor is initiated and how convincing it is. Furthermore the underlying network structure decides how fast the information can spread and how many people are reached. More generally, we can speak of diffusion of information in networks. The analysis of these diffusion processes can be useful for viral marketing, e.g., to target a few influential people to initiate marketing campaigns. A company may wish to distribute the rumor of a new product via the most influential individuals in popular social networks such as Facebook. A second company might want to introduce a competing product and has hence to select where to seed the information to be disseminated. Rumor spreading is quite similar to our ice cream stand problem.

More formally, we may study propagation problems in graphs. Given a graph, and two players. Let the first player choose a seed node  $u_1$ ; afterwards let the second player choose a seed node  $u_2$ , with  $u_2 \neq u_1$ . The goal of the game is to maximize the number of nodes that are closer to one's own seed node.

In many graphs it is an advantage to choose first. In a star graph for instance the first player can choose the center node of the star, controlling all but one

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BIBLIOGRAPHY 99	node. In some other graphs, the second player can at least score even. But is there a graph where the second player has an advantage? <b>Theorem 9.10</b> . In a two alonger sympor nome where both players select one node	to initiate their rumor in the graph, the first player does not always win. The first player does not always win.	Tryp, respectively the decision for an example where the second pays with energy less of the decision the first player. If the first player. He here the node $x_0$ in the center, the second player can select $x_1$ . Choice $x_1$ will be outwitted by $x_2$ , and $x_2$ itself can be answered by $z_1$ . All other strategies are either symmetric,	or even less promising for the first player.					Figure 9.3: Counter example.		<b>Chapter Notes</b> A simple form of a social network is the famous stable marriage problem [DS62]	in which a stable matching bipartite graph has to be found. There exists a great many of variations which are based on this initial problem, e.g., [KC82, KMV94, EO06, FKPS10, Hoe111. Social networks like Facebook, Twitter and others have	grown very fast in the last years and hence spurred interest to research them. How users influence other users has been studied both from a theoretical point	or year [INALUO] and in practice [CHDGLO]. The structure of these networks can be measured and studied [MMG <sup>+</sup> 07]. More than half of the users in social networks share more information than they expect to [LGKM11].	The small world phenomenon that we presented in this chapter is analyzed by Kleinberg [Kle00]. A general overview is in [DJ10]. This chapter has been written in collaboration with Michael Kuhn.	Bibliography	