FS 2008 Prof. R. Wattenhofer / Dr. F. Kuhn / T. Locher / Y. A. Oswald / C. Lenzen

Principles of Distributed Computing
Exercise 4: Sample Solution

1 Deterministic Maximal Independent Set

a) Consider the graph consisting of a connected chain of k nodes $v_1, \ldots, v_k$. We add $i-1$ additional edges leading to $i-1$ additional nodes at node v_i for all $i \in \{1, \ldots, k-1\}$ and k additional edges to k additional nodes at node v_k. The degree $\delta(v_1)$ of v_1 is 1 and for all other nodes $v_i \in \{2, \ldots, k\}$ we have that $\delta(v_i) = i + 1$. All additional nodes have degree 1.

In the first round, all nodes except v_k have a neighbor with a larger degree, thus only v_k joins the MIS. Afterwards, only v_{k-1} can decide and so on. Thus, after k time all nodes $v_1, \ldots, v_k$ and also the additional nodes have decided to join or not to join the MIS.

The number of nodes in this graph is

$$n = k + \sum_{i=1}^{k}(i-1) + 1 = 1 + \sum_{i=1}^{k} i = 1 + \frac{k(k+1)}{2} \leq \frac{(k+1)^2}{2}.$$

The time complexity is thus $k \geq \sqrt{2n} - 1 \in \Omega(n)$.

b) Instead of a chain, we now consider a ring of k nodes $v_1, \ldots, v_k$. We use $k-1$ additional nodes $u_1, \ldots, u_{k-1}$ to increase the degrees of the nodes v_i: There is an edge $\{v_i, u_j\}$ from all nodes v_i to all nodes u_j, where $j \in \{1, k-i\}$. It is easy to see that the degree $\delta(v_i)$ of node v_i is $k + 2 - i$, and that $\delta(u_j) = k - j$.

In the first round, only v_1 joins the MIS. This means that all nodes $u_1, \ldots, u_{k-1}$ and also v_2 and v_k can no longer join the MIS. Thus, in the second round, all these nodes broadcast their decision to all their neighbors that they will not join the MIS. In the third round, only v_3 decides to join the MIS because all other undecided nodes have an undecided neighbor with a larger degree. Subsequently, only v_4 decides (not to join the MIS) in round 4. Repeating this argument, we get that the last node v_{k-1} makes its decision not before round $k-1$. Since $n = k + (k-1) < 2k$, the time complexity is thus $k > \frac{n}{2} \in \Omega(n)$.

2 Randomized Maximal Independent Set

Consider the following graph consisting of two components: Node v is attached to nodes $v_1, \ldots, v_{n/2-1}$. The second component is a chain of nodes $u_1, \ldots, u_{n/2}$. The two components are connected at the nodes v and u_1, i.e., there is an edge between v and u_1. Thus, the graph consists of a star (with center v) and an attached chain of $\frac{n}{2}$ nodes.

The maximum degree is $\Delta = \delta(v) = \frac{n}{2}$. Thus, all the nodes in the chain (except $u_{n/2}$) mark themselves with probability $\frac{2}{2n/2} = \frac{2}{n}$. As the length of the chain is $\frac{n}{2}$, only one node in the chain marks itself in expectation. Its neighbors can also be removed from the graph, as they can no longer join the MIS. Hence, the chain is "cut" into two subchains. In order to remove 3 nodes again, more than 1 round is required in the next step, as less than 1 node marks itself in expectation in every round. Thus, more than $\frac{n}{6}$ rounds are required to remove all the nodes in the chain. The time complexity is thus $\Omega(n)$.