A New Mechanism for the Free-rider Problem

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ABSTRACT

The free-rider problem arises in the provisioning of public resources, when users of the resource have to contribute towards the cost of production. Selfish users may have a tendency to misrepresent preferences – so as to minimize individual contributions – leading to inefficient levels of production of the resource. Groves and Loeb formulated a classic model capturing this problem, and proposed (what later came to be known as) the VCG mechanism as a solution. However, in the presence of heterogeneous users and communication constraints, or in decentralized settings, implementing this mechanism places an unrealistic communication burden. In this paper we propose a class of alternative mechanisms for the same problem as considered by Groves and Loeb, but with the added constraint of severely limited communication between users and the provisioning authority. When these mechanisms are used, efficient production is ensured as a Nash equilibrium outcome, for a broad class of users. Furthermore, a natural bid update strategy is shown to globally converge to efficient Nash equilibria. An extension to multiple public goods with inter-related valuations is also presented.

Categories and Subject Descriptors

G.1.6 [**Optimization**] Mechansim Design, Non-cooperative Games

General Terms

Economics, Algorithms, Theory

1. INTRODUCTION

This paper proposes a new class of mechanisms for addressing the free-rider problem that arises in the production of public goods. By *public good* we refer to a resource whose

usage is non-exclusionary: it can be used simultaneously and equally by all users. This is in contrast to a *private good*, which has to be divided up among the users, each of whom has exclusive access to its portion after the auction. Common examples of public goods in everyday life are television / radio broadcasts, weather reports and public works such as libraries.

In proposing the mechanisms described in this paper we are motivated by public goods in modern communication and computation systems. Consider for example a large distributed database, containing information available to all users, without exclusion. Each user contributes towards the building / maintenance of this database, either in direct monetary terms or through contributed storage resources. Since the information in the database is assumed to be freely available to all users, each user has an incentive to minimize the amount of resources it contributes. However, if every user acts according to these selfish considerations, the net result could be a possibly severe under-provisioning of the resource. This is the classic "free-rider problem:" improper provisioning of a public good – the database – due to selfish behavior. Other examples of public resources are community wireless data access, and file distribution and storage in peer-to-peer networks.

Mechanisms for the production of public goods proceed as follows. Users are asked to submit bids to the producer. Based on the received bids the producer then decides, according to a pre-specified and globally known rule, the quantity of the public good to be produced and the contributions to be made by each of the users. Groves and Loeb [1] proposed a generic model capturing the free-rider problem in the production of a real-valued amount of a public good. The mechanism they proposed for solving the problem was one of the earliest instances of what later came to be known as the general class of VCG mechanisms. This paper proposes alternative mechanism designs for the same resource allocation problem as formulated in [1].

It is well known (see e.g. [2]) that VCG mechanisms are the only ones that ensure efficient production as dominant strategy outcomes in a wide variety of resource allocation problems. It is also increasingly apparent that in many settings the implementation of VCG mechanisms places a heavy communication and computational demand on the auctioneer and agents, to the extent that they are deemed infeasible to implement. Another criticism of the VCG mechanism is that it asks for detailed private information, namely the entire set of user preferences, to be made public for the purposes of resource allocation. Even when bids may be

submitted anonymously, users may be unwilling or unable to completely reveal their preferences.

In this paper we consider the same problem as was considered in [1], but add a severe communication constraint. Specifically, we require that each user's bid be a *single* real number. This is in contrast to the VCG implementation, which asks that the bid be an entire real-valued *function*. Since dominant strategy equilibria are unreasonable to expect in this setting, we settle for Nash strategies as the equilibrium concept. We propose mechanisms that result in the production of an optimal quantity of the public good at any Nash equilibrium. Furthermore, Nash equilibria are shown to always exist, and are unique if there is a unique optimal quantity. Revelation of single-valued bids implies that it is not possible to infer a user's private valuation information from its bid.

Such a mechanism design immediately raises the question of price discovery: how do users know / arrive at a Nash equilibrium? This is not a concern for VCG mechanisms as users are assumed to know their own value functions. For the mechanisms in this paper, myopic best response adjustments to bids in continuous time result in global convergence to Nash equilibria. Furthermore, these updates are easy to compute and need very little information – which can be provided by the mechanism designer – about the rest of the market.

VCG mechanisms are individually rational because when bidding optimally each user can ensure that its payment does not exceed the value it obtains from the good's production. In the mechanism proposed in this paper, the payment made by a user will always be less than its single-valued bid. This is true at all times, not just at the final equilibrium. Also, at Nash equilibrium, no user's payment will exceed the value it obtains from the good's production.

Allocation of *private* goods based on single-valued bids, as well as the dynamics of convergence to equilibria, has received attention previously [3, 4, 5, 6, 7]. The problem of provisioning a public good differs from these in the sense that the allocation decision is one-dimensional, namely the real-valued quantity to be produced, while in the private good case the allocation is a multi-dimensional vector of each user's quantity.

For public goods, when improperly designed mechanisms are used, inefficient production may occur as an equilibrium outcome. In Section 2 we give an example of a simple intuitive mechanism and show that inefficiencies in production may be quite severe even for reasonable scenarios of user value functions. In Section 3 we describe one example of the class of new mechanism we propose in this paper, and prove the existence, uniqueness, and optimality results for Nash equilibria in the resulting game. In Section 4 we show that myopic gradient ascent updates in continuous time result in global convergence to a Nash equilibrium. In Section 5 we extend the example mechanism to the case when there are multiple public goods and users with joint value functions. We present the full class of mechanisms in Section 6, and conclude with some discussion in Section 7.

2. SYSTEM MODEL

A certain quantity $Q \in \mathbb{R}_{++}$ of a public good has to be produced by a producer, where $\mathbb{R}_{++}$ is the set of strictly positive reals. The producer can produce a quantity Q at cost $C(Q)$. Once produced, it is available to n users, where

$n \geq 2$. Each user i obtains a value $U_i(Q)$ from the good's production, and contributes a payment p_i towards its production. It is assumed that $C(Q)$ is strictly increasing and convex and each $U_i(Q)$ is strictly increasing and concave, and all functions are continuously differentiable. This is the same as the model in Groves and Loeb [1]. [1]

A quantity Q^* is said to be *efficient* if producing that quantity maximizes the net social benefit:

$$\sum_i U_i(Q^*) - C(Q^*) \geq \sum_i U_i(Q) - C(Q)$$

for all $Q \in \mathbb{R}_{++}$. If a quantity Q does not satisfy the above requirement, it is inefficient. It is assumed that there exists some finite $Q^* > 0$ that is efficient. Concavity implies that the efficiency of Q^* is characterized by the first-order conditions.

LEMMA 2.1. *A quantity Q^* is socially optimal iff* $\sum_i U_i'(Q^*) = C'(Q^*)$.

Any mechanism for the production of the good proceeds as follows. First, each user i is asked to submit a *bid b_i*. Then, the producer maps the vector of bids $\mathbf{b}$ into a produced quantity $f(\mathbf{b})$ and a payment $p_i(\mathbf{b})$ for each user i. We will call f the *production function* and the p_is the *payment functions*. The production and payment functions are known by the users in advance, i.e. before they submit their bids. Specifying the space of allowed bids and the production and payment functions specifies the mechanism.

One example of such a mechanism is the classical VCG mechanism. A VCG mechanism requires users to submit bids that are functions on $\mathbb{R}_+$. Given these bid functions b_i, the production function is

$$f^{VCG}(\mathbf{b}) = \arg \max_{Q \geq 0} \sum_i b_i(Q) - C(Q)$$

while the payment function for user i is

$$p_i^{VCG}(\mathbf{b}) = \left(\max_{Q \geq 0} \sum_{j \neq i} b_j(Q) - C(Q) \right)$$
$$- \left(\sum_{j \neq i} b_j(f^{VCG}(\mathbf{b})) - C(f^{VCG}(\mathbf{b})) \right)$$

Given a mechanism and bid vector $\mathbf{b}$, the *net reward* of user i is given by

$$R_i(\mathbf{b}) = U_i(f(\mathbf{b})) - p_i(\mathbf{b}) \qquad (1)$$

Given the mechanism, the users play a non-zero-sum non-cooperative game, with each user trying to maximize its own net reward.

As an example of a mechanism susceptible to the free-rider problem, consider the *pay as bid* mechanism where user payments are the bids – $p_i(\mathbf{b}) = b_i$ – and the production function is the one that balances the budget – $f(\mathbf{b}) = X(B)$ where $X = C^{-1}$ is the inverse of the cost function and $B = \sum_i b_i$ is the total of all bids (and payments).

For this mechanism it can be seen that the first-order necessary condition for a bid vector $\widetilde{\mathbf{b}}$ to be a Nash equilibrium

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[1] Except that in [1] it is assumed that $Q \geq 0$ and $C(Q) = pQ$ for some $p > 0$. Also the U_i functions need not be differentiable.

is that

$$U_i'(X(\widetilde{B})) \ \leq \ C'(X(\widetilde{B}))$$

for *each* user i, with equality if $\widetilde{b}_i > 0$. It is easy to see from Lemma 2.1 that there will not be an efficient Nash equilibrium when more than two users are present, for *any* set of value functions U_i and cost function C.

3. A NEW MECHANISM

In this section we present an example from the new class of mechanisms that ensure socially optimal production. To do so we need to specify the space of allowable bid vectors, the production function f, and the payment functions p_i.

Each user's bid is a strictly positive real number: $b_i \in \mathbb{R}_{++}$. Given the cost function $C(Q)$, define its inverse function X. This can be done since C in strictly increasing. Thus $X(C(Q)) = Q$ for all $Q \in \mathbb{R}_{++}$. Also, X is increasing, concave and differentiable, with $X'(C(Q)) = \frac{1}{C'(Q)}$.

Given the vector of bids $\underline{b}$, denote the total of all bids by $B = \sum_i b_i$. We propose the following production function:

$$f^*(\underline{b}) \ = \ X(B) \qquad (2)$$

For each user i, denote the total bid of users other than i by $B_{-i} = \sum_{j \neq i} b_j$. We propose the following payment functions

$$p_i^*(\underline{b}) \ = \ b_i - B_{-i} \log\left(1 + \frac{b_i}{B_{-i}}\right) \qquad (3)$$

The mechanism is thus fully specified.

LEMMA 3.1. *The reward function $R_i(b_i, B_{-i})$ is concave in b_i for all fixed values of b_j, $j \neq i$.*

LEMMA 3.2. $0 < p_i(b_i, B_{-i}) < b_i$ *for all $\underline{b}$ and i: a user is never asked to pay more than its bid.*

Since each user's reward function (1) is concave in its own bid, the simultaneous satisfaction of the following first-order conditions by all users at a bid vector $\underline{\widetilde{b}}$ is necessary and sufficient for $\underline{\widetilde{b}}$ to be a Nash equilibrium:

$$U_i'(X(\widetilde{B})).X'(\widetilde{B}) \ - \ 1 + \frac{\widetilde{B}_{-i}}{\widetilde{b}_i + \widetilde{B}_{-i}} \ = \ 0 \ \text{ for all } i$$

If $\widetilde{Q} = X(\widetilde{B})$ then $X'(\widetilde{B}) = \frac{1}{C'(\widetilde{Q})}$ and so the above conditions can be rewritten as

$$U_i'(\widetilde{Q}) \ = \ \frac{\widetilde{b}_i}{\widetilde{B}} C'(\widetilde{Q}) \ \text{ for all } i \qquad (4)$$

Thus we see that $\sum_i U_i'(\widetilde{Q}) = C'(\widetilde{Q})$ at Nash equilibrium. By Lemma 2.1 this relation implies that $\widetilde{Q}$ will be efficient. We state the theorem formally below, and prove it in the appendix.

THEOREM 3.1. *For the public good model described in the previous section if the mechanism (f^*, p_i^*) is used, there is a one-to-one correspondence between the set of efficient quantities and the set of Nash equilibria for the game. Also, at any of these Nash equilibria the corresponding efficient quantity is provisioned.*

Note: Rosen's theorem [8] cannot be directly used to show the existence of Nash equilibria in this game since the users' strategy spaces are not closed.

It is clear that the mechanism is not budget-balanced. However, it is possible to bound the subsidy $B - \sum_i p_i(\underline{b})$ as a fraction of the total cost.

PROPOSITION 3.1. *When n users are present,*

$$\frac{B - \sum_i p_i(\underline{b})}{B} \ \leq \ (n-1)\log\frac{n}{n-1}$$

This is tight when the n bids are equal.

4. DYNAMICS

The above section shows that for the mechanism presented, Nash equilibria always exist and are efficient. However, users still have to find out these Nash equilibria. In this section we show that if users follow a natural bid update strategy, then the vector of bids converges to a Nash equilibrium from any valid initial condition.

Specifically, consider the user update rule when each user attempts gradient ascent, in continuous time, of its reward function (1):

$$\begin{aligned} \frac{d}{dt}b_i \ &= \ \frac{\partial}{\partial b_i} R_i(b_i, B_{-i}) \\ &= \ U_i'(X(B)).X'(B) - \frac{b_i}{B} \qquad (5) \end{aligned}$$

To follow this bid update procedure at a given time, the user only needs to know the amount currently provisioned, the cost function's derivative and the total of all the users' bids. The user does not need detailed information about what each user's bid is, or even how many users are present.

THEOREM 4.1. *For the bid updates given by (5), the vector of bids will converge to a Nash equilibrium from any initial condition having $b_i > 0$ for at least two users i.*

If the unique optimal decision is zero production, i.e. if

$$\sum_i U_i(0) - C(0) \ > \ \sum_i U_i(Q) - C(Q)$$

for all $Q > 0$, then the above dynamics will result in $B \to 0$. Hence production will be efficient in the limit.

5. MULTIPLE GOODS

The mechanism presented above has a natural extension to the case when there are multiple public goods to be produced for users who have joint valuation functions. In this section we show that if the production costs are decoupled, then using the mechanism proposed in this paper separately for each good results in efficient joint provisioning of all goods.

Efficient production has to be achieved of M public goods. Denote the vector of quantities by $Q = [Q_1, \ldots, Q_M]$. Each user has a value function $U_i(Q)$, which is assumed to be jointly continuous, differentiable, concave and strictly increasing [2] in each coordinate. The production of each good incurs a cost, as specified by the cost functions $C_m(Q_m)$ for

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[2] The strictly increasing requirement for U_i can be relaxed somewhat, but we will not discuss it here for brevity

$1 \leq m \leq M$. Each cost function C_m is assumed to be convex, strictly increasing and differentiable. $Q \in \mathbb{R}_{++}^M$ means that each coordinate is strictly positive: $Q_m \in \mathbb{R}_{++}$ for all m.

A vector of quantities Q^* is said to be *efficient* if it maximizes the net social benefit:

$$\sum_i U_i(Q^*) - \sum_m C_m(Q_m^*) \geq \sum_i U_i(Q) - \sum_m C_m(Q_m)$$

for all $Q \in \mathbb{R}_{++}^M$. It is assumed that there is at least one efficient $Q^* \in \mathbb{R}_{++}^M$ in which each quantity is finite.

With these assumptions, running a separate market for each good results in efficient production. Each user i now submits a vector of bids $b_i = [b_i^1 \dots b_i^M]$: thus each bid is an M-dimensional vector $b_i \in \mathbb{R}_{++}^M$. As before define, for each good m, the bid sums $B^m \triangleq \sum_i b_i^m$ and $B_{-i}^m \triangleq \sum_{j \neq i} b_j^m$. Let X_m be the inverse function of C_m as in the single good case. For notational brevity, denote the vector of total bids by $B = [B^1, \dots, B^M]$ and the vector production function by X. Thus $X(B)$ stands for the vector of quantities $[X_1(B^1), \dots, X_M(B^M)]$. Also, $B_{-i} = [B_{-i}^1, \dots, B_{-i}^M]$.

Consider the mechanism that, given all the bids, produces quantity $X_m(B^m)$ of each good m and charges user i an amount $\sum_m p_i^m$, where

$$p_i^m(b_i^m, B_{-i}^m) = b_i^m - B_{-i}^m \log\left(1 + \frac{b_i^m}{B_{-i}^m}\right)$$

is the payment user i makes towards the provisioning of good m. The level of production of each good is thus a local decision, with the users balancing payments across goods so as to maximize their net reward. For the mechanism as described, this is given by

$$R_i(b_i, B_{-i}) = U_i(X(b_i + B_{-i})) - \sum_m p_i^m(b_i^m, B_{-i}^m)$$

Bid vectors $\widetilde{b}_1, \dots, \widetilde{b}_n$ are a Nash equilibrium if

$$R_i(\widetilde{b}_i, \widetilde{B}_{-i}) \geq R_i(b_i, \widetilde{B}_{-i}) \text{ for all } b_i \in \mathbb{R}_{++}^M$$

As in the single good case, efficient allocations and Nash equilibria are fully characterized by first-order conditions. Thus Q^* is optimal if and only if

$$\sum_i \frac{\partial}{\partial Q_m} U_i(Q^*) = C_m'(Q_m^*) \text{ for each } m$$

Bid vectors $\widetilde{b}_1, \dots, \widetilde{b}_n$ are a Nash equilibrium if and only if

$$\frac{\partial}{\partial Q_m} U_i(\widetilde{Q}) = \frac{\widetilde{b}_i^m}{\widetilde{B}^m} C_m'(\widetilde{Q}_m) \text{ for all } i \text{ and } m$$

where $\widetilde{Q} = X(\widetilde{B})$. This is the multiple-goods analogue of relation (4), and we can prove the existence and efficiency of Nash equilibria for the multiple goods case in the same way as was done for a single good. We state this as a theorem below and omit the proof.

THEOREM 5.1. *Consider the model with multiple public goods described in this section with the mechanism (f^*, p_i^*) used for the provisioning of each good. Then there is a one-to-one correspondence between the set of efficient quantity vectors and the set of Nash equilibria for the game. Also, at any of these Nash equilibria the corresponding efficient quantity vector is provisioned.*

As in the single good case, if each user updates its bid vector according to gradient ascent in continuous time then there is global convergence to an efficient Nash equilibrium. The update equations are now given by the gradient

$$\frac{d}{dt} b_i = \nabla_{b_i} \left(U_i(X(B)) - \sum_m p_i^m(b_i^m, B_{-i}^m) \right) \quad (6)$$

which is the same as

$$\frac{d}{dt} b_i^m = \left(\frac{\partial}{\partial Q_m} U_i(X(B)) \right) X_m'(B^m) - \frac{b_i^m}{B^m}$$

The proof of global convergence is similar to that for a single good. We state the theorem below and omit the proof.

THEOREM 5.2. *For the bid updates given by (6), the vector of bids will converge to a Nash equilibrium from any initial condition where for each good m there are at least two users with strictly positive bids for that good.*

6. A CLASS OF EFFICIENT MECHANISMS

The mechanism presented in this paper is not unique in its efficiency guarantee. In particular, consider a mechanism with real-valued bids such that the production function is $f(\underline{b}) = X(B)$, each payment function $p_i(\underline{b})$ is convex increasing and continuously differentiable in b_i for fixed $\underline{b}_{-i}$, $p_i(0, \underline{b}_{-i}) = 0$, and the vector of payment functions satisfies

$$\sum_i \frac{\partial p_i}{\partial b_i}(\underline{b}) = 1$$

for all $\underline{b} \in \mathbb{R}_{++}^n$. If the existence of Nash equilibria can be demonstrated for such a mechanism, then the Nash equilibria are guaranteed to be efficient. We will refer to any such mechanism as a *mechanism with guaranteed efficiency*. In the following, we present a large class of mechanisms with guaranteed efficiency for which the existence of Nash equilibria can be shown.

Let $\psi(s)$ be a strictly increasing continuous function from $[0, \infty)$ to $[0, \infty)$, with $\psi(0) = 0$. Given a vector $\underline{b}$ of bids, with $b_i \in \mathbb{R}_{++}$, consider the mechanism given by

$$f(\underline{b}) = X(B)$$
$$p_i(\underline{b}) = \int_0^{b_i} \frac{\psi(s)}{\psi(s) + \sum_{j \neq i} \psi(b_j)} ds$$

In terms of this notation, the mechanism given by (2) and (3) corresponds to $\psi(x) = x$.

It can be shown that the analogues of Lemma 3.1, Lemma 3.2, Theorem 3.1, and Theorem 4.1 hold for any mechanism of the kind specified above. Given this class of efficient mechanisms, further optimizations can be performed with revenue as the metric. The following proposition gives an upper bound on revenue for the more general class of mechanisms with guaranteed efficiency.

PROPOSITION 6.1. *Given user value functions U_i and any mechanism with guaranteed efficiency, if $\underline{b}^*$ is a Nash equilibrium at which quantity Q^* is produced, then*

$$\frac{\sum_i p_i(\underline{b}^*)}{B^*} \leq \max_i \frac{U_i'(Q^*)}{C'(Q^*)}$$

Proof of Prop. 6.1:

For each user i,

$$\frac{p_i(\underline{\mathbf{b}}^*)}{b_i^*} \;\leq\; \frac{\partial p_i}{\partial b_i}(\underline{\mathbf{b}}^*) \;=\; \frac{U_i'(Q^*)}{C'(Q^*)}$$

where the inequality holds because $p_i(b_i, \underline{\mathbf{b}}_{-i}^*)$ is a convex increasing function of b_i with $p_i(0, \underline{\mathbf{b}}_{-i}) = 0$, and the equality is the first order necessary condition for b_i^* to be a Nash bid for user i. The proof of the proposition follows immediately from these relations. □

The above upper bound may be quite small if the number of users is large. Note that efficiency requires that $\sum_i U_i'(Q^*) = C'(Q^*)$. So, for example, in the case when users are symmetric this upper bound is $\frac{1}{n}$.

The mechanism designer may be interested in maximizing revenue. Alternatively, it may be the designer's objective to provision optimally, but at minimal cost to the users. The following theorem shows that either objective can be achieved by a suitably designed mechanism from the class described in this section.

THEOREM 6.1. *Let user value functions $U_1 \ldots, U_n$ and production cost function C be such that there is a unique optimal quanitity $Q^* > 0$ costing $B^* = C(Q^*)$. Let $p_i^\alpha(\underline{\mathbf{b}})$ be the payment function for the mechanism having $\psi(s) = 1 - e^{-\alpha s}$, and $p_i^\beta(\underline{\mathbf{b}})$ for the mechanism having $\psi(s) = e^{\beta s}$. Then, if $\underline{\mathbf{b}}^\alpha$ and $\underline{\mathbf{b}}^\beta$ are the corresponding Nash equilibria,*

$$\lim_{\alpha \to \infty} \frac{\sum_i p_i^\alpha(\underline{\mathbf{b}}^\alpha)}{B^*} \;=\; \max_i \frac{U_i'(Q^*)}{C'(Q^*)}$$

$$\lim_{\beta \to \infty} \frac{\sum_i p_i^\beta(\underline{\mathbf{b}}^\beta)}{B^*} \;=\; 0$$

Note however that the convergence of either of the above limits is not uniform in the choice of the U_i value functions.

7. DISCUSSION

This paper proposes a class of mechanisms to alleviate the free-rider problem by ensuring efficiency at Nash equilibria of a static game. It then shows that user bids converge to Nash equilibria globally, provided they use myopic update strategies. Using iterative price and bid update procedures for computationally infeasible problems in auctions and resource allocation have been proposed recently for multi-unit auctions where users have bundle bids [9, 10], as well as in the allocation of divisible goods [6, 7]. All these mechanisms give efficiency and truthful revelation guarantees only when users are assumed to follow myopic best response bid updates. The analysis of user dynamics as repeated games in the true sense is hard. Furthermore, in general, it seems unlikely that the efficiency properties shown for static mechanisms will hold when the dynamics of convergence are repeated games. The issue of dynamics is thus a genuine point of criticism for this approach. In the settings of modern information systems however, two comments can be made to partially address this issue. Firstly, it may be that the mechanism is not an honest auction, but rather an implementable algorithm to find efficient allocations in the presence of communication constraints. Secondly, in large distributed settings, finding a viable alternative to best-response dynamics may be hard.

Appendix

Proof of Theorem 3.1:

Suppose first that $\underline{\widetilde{\mathbf{b}}}$ is a Nash equilibrium, and $\widetilde{Q} = f^*(\underline{\widetilde{\mathbf{b}}})$ is the corresponding quantity that is produced. The equilibrium condition given in (4) holds for $\underline{\widetilde{\mathbf{b}}}$. Summing the conditions in (4) over the set of users yields

$$\sum_i U_i'(\widetilde{Q}) \;=\; C'(\widetilde{Q})$$

By Lemma 2.1, this means that $\widetilde{Q}$ is efficient. Thus efficient quantities are provisioned at Nash equilibria.

For showing the existence of Nash equilibria, we simply turn the above argument around. Let Q^* be efficient – by assumption there exists at least one such quantity that is finite. Define for each user the bid

$$\widetilde{b}_i \;\triangleq\; \frac{U_i'(Q^*)}{C'(Q^*)} \, C(Q^*)$$

Then, by Lemma 2.1, $\sum_i U_i'(Q^*) \;=\; C'(Q^*)$ and hence the total bid satisfies $\widetilde{B} = C(Q^*)$. Thus the above equation can be rewritten as

$$U_i'(Q^*) \;=\; \frac{\widetilde{b}_i}{\widetilde{B}} \, C'(Q^*) \;\; \text{for all } i$$

Since $Q^* = X(\widetilde{B})$, this vector $\underline{\widetilde{\mathbf{b}}}$ of bids defined from Q^* satisfies the necessary and sufficient conditions of (4), and hence is a Nash equilibrium. $\underline{\widetilde{\mathbf{b}}}$ as defined above corresponds to the efficient quantity Q^*. □

Sketch of Proof of Theorem 4.1:

Note that the problem of provisioning an efficient quantity Q^* is equivalent to ensuring that the user bids total to B^* where

$$B^* \;=\; \arg\max_B \; \sum_i U_i(X(B)) \,-\, B$$

Adding (5) over all users i gives the update equation for the total of all the bids:

$$\frac{d}{dt}B \;=\; \sum_i U_i'(X(B)).X'(B) \,-\, 1$$

Note that this is the same as gradient ascent by B towards the optimum of the concave one-dimensional function $\sum_i U_i(X(B)) - B$. Thus individual gradient ascent in continuous time by each user, for its own reward function, results in global gradient ascent by the sum of the bids towards the efficient outcome.

Also, once the sum bid is close to an efficient B^*, the sum of the bids B remains nearly fixed. At this point if the bid of any user is not its Nash bid, it will converge to a small neighborhood of the Nash bid exponentially fast. □

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