Topology Control in Heterogeneous Wireless Networks: Problems and Solutions

Ning Li and Jennifer C. Hou
Department of Computer Science
University of Illinois at Urbana-Champaign
Urbana, IL 61801
{nli,jhou}@cs.uiuc.edu

Abstract— **Previous work on topology control usually assumes homogeneous wireless nodes with uniform transmission ranges. In this paper, we propose two localized topology control algorithms for heterogeneous wireless multi-hop networks with non-uniform transmission ranges: Directed Relative Neighborhood Graph (DRNG) and Directed Local Minimum Spanning Tree (DLMST). In both algorithms, each node selects a set of neighbors based on the locally collected information. We prove that (1) the topologies derived under DRNG and DLMST preserve the network connectivity; (2) the out degree of any node in the resulting topology by DLMST is bounded; while the out degree of nodes in the topology by DRNG is not bounded; and (3) the topologies generated by DRNG and DLMST preserve the network bi-directionality.**

I. Introduction

Energy efficiency [1] and network capacity are perhaps two of the most important issues in wireless ad hoc networks and sensor networks. Topology control algorithms have been proposed to maintain network connectivity while reducing energy consumption and improving network capacity. The key idea to topology control is that, instead of transmitting using the maximal power, nodes in a wireless multi-hop network collaboratively determine their transmission power and define the network topology by forming the proper neighbor relation under certain criteria.

By enabling wireless nodes to use adequate transmission power (which is usually much smaller than the maximal transmission power), topology control can not only save energy and prolong network lifetime, but also improve spatial reuse (and hence the network capacity) [2] and mitigate the MAC-level medium contention [3]. Several topology control algorithms [3]–[10] have been proposed to create power-efficient network topology in wireless multi-hop networks with limited mobility (a summary is given in Section III). However, most of them assume homogeneous wireless nodes with uniform transmission ranges (except [4]).

The assumption of homogeneous nodes does not always hold in practice since even devices of the same type may have slightly different maximal transmission power. There also exist heterogeneous wireless networks in which devices have dramatically different capabilities, for instance, the communication network in the Future Combat System which involves wireless devices on soldiers, vehicles and UAVs. As will be exemplified in Section III, most existing algorithms cannot be directly applied to heterogeneous wireless multi-hop networks in which the transmission range of each node may be different.

To the best of our knowledge, this paper is the first effort to address the connectivity and bi-directionality issue in the heterogeneous wireless networks.

In this paper, we propose two localized topology control algorithms for heterogeneous wireless multi-hop networks with non-uniform transmission ranges: Directed Relative Neighborhood Graph (DRNG) and Directed Local Minimum Spanning Tree (DLMST). In both algorithms, the topology is constructed by having each node build its neighbor set and adjust its transmission power based on the locally collected information. We are able to prove that (1) the topology derived under both DRNG and DLMST preserves network connectivity, i.e., if the original topology generated by having every node use its maximal transmission power is strongly connected, then the topologies generated by both DRNG and DLMST are also strongly connected; (2) the out degree of any node in the topology by DLMST is bounded, while the out degree of nodes in the topology by DRNG may be unbounded; and (3) the topology generated by DRNG and DLMST preserves network bi-directionality, i.e., if the original topology by having every node use its maximal transmission power is bi-directional, then the topology generated by either DRNG or DLMST is also bi-directional after some simple operations.

Simulation results indicate that compared with the other known topology control algorithms that can be applied to heterogeneous networks, DRNG and DLMST have smaller average node degree (both logical and physical) and smaller average link length. The former reduces the MAC-level contention, while the latter implies a small transmission power needed to maintain connectivity.

The rest of the paper is organized as follows. In Section II, we give the network model. In Section III, we summarize previous work on topology control, and give examples to show why existing algorithms cannot be directly applied to heterogeneous networks. Following that, we present both the DRNG and DLMST algorithms in Section IV, and prove several of their useful properties in Section V. Finally, we evaluate the performance of the proposed algorithms in Section VI, and conclude the paper in Section VII.

II. Network Model

Consider a set of nodes, $V = \{v_1, v_2, \ldots, v_n\}$, which are randomly distributed in the 2-D plane. Let r_{v_i} be the maximal transmission range of v_i. In a heterogeneous network, the maximal transmission ranges of all nodes may not be the same. Let $r_{min} = \min_{v \in V}\{r_v\}$ and $r_{max} = \max_{v \in V}\{r_v\}$. We denote the network topology generated by having each node use its own maximal transmission power as a simple directed graph $G = (V(G), E(G))$, where $E(G) = \{(u, v) : d(u, v) \leq r_u, u, v \in V(G)\}$ is the edge set of G and $d(u, v)$ is the Euclidean distance between node u and node v. Note that (u, v) is an ordered pair representing an edge from node u to node v. A unique *id* (such as an IP/MAC address) is assigned to each node. Here we let $id(v_i) = i$ for simplicity.

We assume that the wireless channel is symmetric [1] and obstacle-free, and each node is equipped with the capability to gather its location information via, for example, GPS for outdoor applications and pseudolite [11] for indoor applications, and many other lightweight localization techniques for wireless networks (see [12] for a summary).

Before delving into the technical discussion and algorithm description, we give the definition of several terms that will be used throughout the paper.

Definition 1 (Reachable Neighborhood): The reachable neighborhood N_u^R is the set of nodes that node u can reach using its maximal transmission power, i.e., $N_u^R = \{v \in V(G) : d(u, v) \leq r_u\}$. For each node $u \in V(G)$, let $G_u^R = (V(G_u^R), E(G_u^R))$ be an induced subgraph of G such that $V(G_u^R) = N_u^R$.

Definition 2 (Weight Function): Given two edges $(u_1, v_1), (u_2, v_2) \in E$ and the Euclidean distance function $d(\cdot, \cdot)$, weight function $w : E \mapsto R$ satisfies:

$$\begin{aligned} & w(u_1, v_1) > w(u_2, v_2) \\ \Leftrightarrow \quad & d(u_1, v_1) > d(u_2, v_2) \\ \text{or} \quad & (d(u_1, v_1) = d(u_2, v_2) \\ & \quad \&\& \max\{id(u_1), id(v_1)\} > \max\{id(u_2), id(v_2)\}) \\ \text{or} \quad & (d(u_1, v_1) = d(u_2, v_2) \\ & \quad \&\& \max\{id(u_1), id(v_1)\} = \max\{id(u_2), id(v_2)\} \\ & \quad \&\& \min\{id(u_1), id(v_1)\} > \min\{id(u_2), id(v_2)\}). \end{aligned}$$

This weight function ensures that two edges with different end nodes have different weights. Note that $w(u, v) = w(v, u)$.

Definition 3 (Neighbor Set): Node v is a *neighbor* of node u under an algorithm A, denoted $u \xrightarrow{A} v$, if and only if there exists an edge (u, v) in the topology generated by the algorithm. In particular, we use $u \rightarrow v$ to denote the neighbor relation in G. $u \overset{A}{\Leftrightarrow} v$ if and only if $u \xrightarrow{A} v$ and $v \xrightarrow{A} u$. The *Neighbor Set* of node u is $N_A(u) = \{v \in V(G) : u \xrightarrow{A} v\}$.

Definition 4 (Topology): The topology generated by an algorithm A is a directed graph $G_A = (E(G_A), V(G_A))$, where $V(G_A) = V(G)$, $E(G_A) = \{(u, v) : u \xrightarrow{A} v, u, v \in V(G_A)\}$.

[1] By symmetric we mean that both the sender and the receiver should observe the same channel properties such as interference, path loss, and fading.

Definition 5 (Radius): The radius, r_u, of node u is defined as the distance between node u and its farthest neighbor (in terms of Euclidean distance), i.e, $r_u = \max_{v \in N_A(u)}\{d(u, v)\}$.

Definition 6 (Connectivity): For any topology generated by an algorithm A, node u is said to be *connected to* node v (denoted $u \Rightarrow v$) if there exists a path ($p_0 = u, p_1, \ldots, p_{m-1}, p_m = v$) such that $p_i \xrightarrow{A} p_{i+1}, i = 0, 1, \ldots, m-1$, where $p_k \in V(G_A), k = 0, 1, \ldots, m$. It follows that $u \Rightarrow v$ if $u \Rightarrow p$ and $p \Rightarrow v$ for some $p \in V(G_A)$.

Definition 7 (Bi-Directionality): A topology generated by an algorithm A is *bi-directional*, if for any two nodes $u, v \in V(G_A)$, $u \in N_A(v)$ implies $v \in N_A(u)$.

Definition 8 (Bi-Directional Connectivity): For any topology generated by an algorithm A, node u is said to be *bi-directionally connected to* node v (denoted $u \Leftrightarrow v$) if there exists a path ($p_0 = u, p_1, \ldots, p_{m-1}, p_m = v$) such that $p_i \overset{A}{\Leftrightarrow} p_{i+1}, i = 0, 1, \ldots, m-1$, where $p_k \in V(G_A), k = 0, 1, \ldots, m$. It follows that $u \Leftrightarrow v$ if $u \Leftrightarrow p$ and $p \Leftrightarrow v$ for some $p \in V(G_A)$.

Deriving network topology consisting of only bi-directional links facilitates link level acknowledgment, which is a critical operation for packet transmissions and retransmissions over unreliable wireless media. Bi-directionality is also important in floor acquisition mechanisms such as the RTS/CTS mechanism in IEEE 802.11.

Definition 9 (Addition and Removal): The *Addition* operation is to add an extra edge (v, u) into G_A if $(u, v) \in E(G_A)$, $(v, u) \notin E(G_A)$, and $d(u, v) \leq r_v$. The *Removal* operation is to delete any edge $(u, v) \in E(G_A)$ if $(v, u) \notin E(G_A)$.

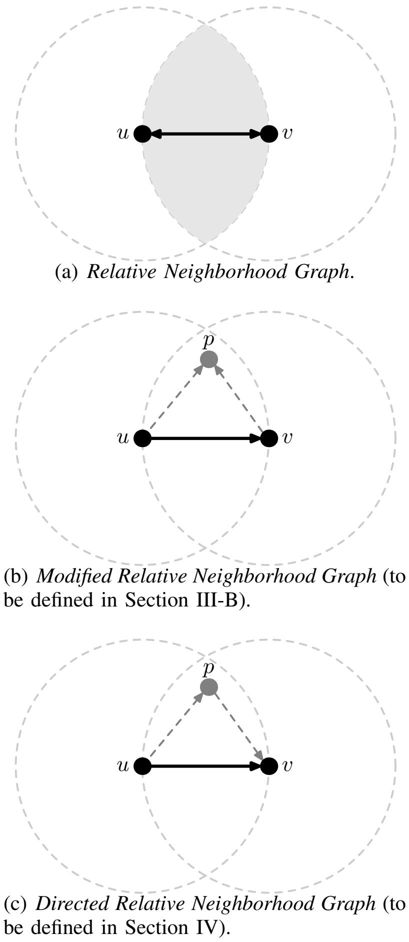
Both the *Addition* and *Removal* operations attempt to create a bi-directional topology by removing uni-directional edges or converting uni-directional edges into bi-directional. The resulting topology after *Removal* is alway bi-directional, although it may be disconnected. The resulting topology after *Addition* is not necessarily bi-directional, as it essentially tries to increases the transmission power of a node v to a level that may be beyond its capability.

III. Related Work and Why They Cannot be Directly Applied to Heterogeneous Networks

Several topology control algorithms [3]–[10] have been proposed. In this section, we first summarize these algorithm and then give examples on why they cannot be directly applied to heterogeneous networks.

A. Related Work

Rodoplu *et al.* [4] (denoted R&M) introduced the notion of *relay region* and *enclosure* for the purpose of power control. Instead of transmitting directly, a node chooses to relay through other nodes if less power is consumed. It is shown that the network is strongly connected if every node maintains links with the nodes in its enclosure and the resulting topology is a minimum power topology. The major drawback is that it requires an explicit propagation channel model to compute the relay region. (In the simulation study presented in Section VI, we assume that the two-ray ground model is



(a) *Relative Neighborhood Graph.*

(b) *Modified Relative Neighborhood Graph* (to be defined in Section III-B).

(c) *Directed Relative Neighborhood Graph* (to be defined in Section IV).

Fig. 1. The definition of the *Directed Relative Neighborhood Graph*.

used.) Also, it assumes there is only one data sink (destination) in the network.

Ramanathan *et al.* [5] presented two centralized algorithms to minimize the maximal power used per node while maintaining the (bi)connectivity of the network. They introduced two distributed heuristics for mobile networks. Both centralized algorithms require global information, and thus cannot be directly deployed in the case of mobility. On the other hand, the proposed heuristics cannot guarantee the preservation of the network connectivity.

COMPOW [3] and *CLUSTERPOW* [7] are approaches implemented in the network layer. Both hinge on the idea that if each node uses the smallest common power required to maintain network connectivity, the traffic carrying capacity of the entire network is maximized, the battery life is extended, and the MAC-level contention is mitigated. The major drawback is its significant message overhead, since each node has to run multiple daemons, each of which has to exchange link state information with their counterparts at other nodes.

CBTC(α) [6] is a two-phase algorithm in which each node finds the minimum power p such that some node can be reached in every cone of degree α. The algorithm has been proved to preserve network connectivity if $\alpha < 5\pi/6$. Several optimization methods (that are applied after the topology is derived under the base algorithm) are also discussed to further reduce the transmitting power.

To facilitate the following discussion, the definition of the *Relative Neighborhood Graph* (RNG) is given below.

Definition 10 (Neighbor Relation in RNG): For RNG [13], [14], $u \stackrel{RNG}{\longleftrightarrow} v$ if and only if there does not exist a third node p such that $w(u,p) < w(u,v)$ and $w(p,v) < w(u,v)$. Or equivalently, there is no node inside the shaded area in Fig. 1(a).

Borbash and Jennings [8] proposed to use RNG for the topology initialization of wireless networks. Based on the local knowledge, each node makes decisions to derive the network topology based on RNG. The network topology thus derived has been reported to exhibit good overall performance in terms of power usage, low interference, and reliability.

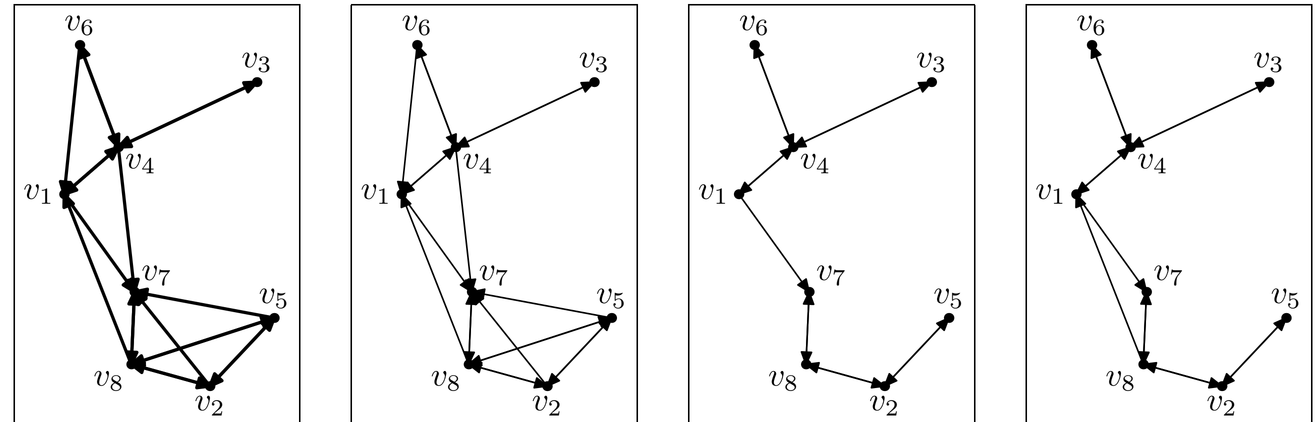
Li *et al.* [9] presented the Localized Delaunay Triangulationa, a localized protocol that constructs a planar spanner of the *Unit Disk Graph* (UDG). The topology contains all edges that are both in the unit-disk graph and the Delaunay triangulation of all nodes. It is proved that the shortest path in this topology between any two nodes u and v is at most a constant factor of the shortest path connecting u and v in UDG. However, the notion of UDG and Delaunay triangulation cannot be directly extended to heterogeneous networks.

In [10], we proposed LMST (Local Minimum Spanning Tree) for topology control in homogeneous wireless multi-hop networks. In this algorithm, each node builds its local minimum spanning tree independently and only keeps on-tree nodes that are one-hop away as its neighbors in the final topology. It is proved that (1) the topology derived under LMST preserves the network connectivity; (2) the node degree of any node in the resulting topology is bounded by 6; and (3) the topology can be transformed into one with bi-directional links (without impairing the network connectivity) after removal of all uni-directional links. Simulation results show that LMST can increase the network capacity as well as reduce the energy consumption.

Instead of adjusting the transmission power of individual devices, there also exist other approaches to generate power-efficient topology. By following a probabilistic approach, Santi *et al.* derived the suitable common transmission range which preserves network connectivity, and established the lower and upper bounds on the probability of connectedness [15]. In [16], a "backbone protocol" is proposed to manage large wireless ad hoc networks, in which a small subset of nodes is selected to construct the backbone. In [17], a method of calculating the power-aware connected dominating sets was proposed to establish an underlying topology for the network.

B. Why Existing Algorithms Cannot be Directly Applied to Heterogeneous Networks

All topology control algorithms, except [4], assume homogeneous wireless nodes with uniform transmission ranges. When directly applied to heterogeneous networks, these algorithms may render disconnectivity. In this subsection, we



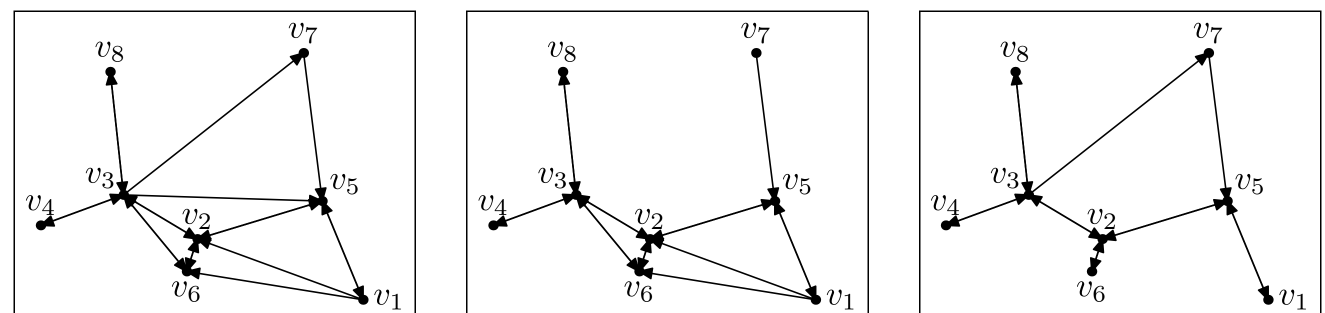
(a) Original topology (without topology control) is strongly connected.

(b) Topology by $CBTC(\frac{5}{6}\pi)$ without optimization is strongly connected.

(c) Topology by CBTC after optimization is not strongly connected: there is no path from v_7 to v_1.

(d) Topology by DLMST is strongly connected.

Fig. 2. An example that shows the optimization in $CBTC(\frac{5}{6}\pi)$ may lead to disconnectivity. An arrow from node v_i to node v_j indicates that v_i can reach v_j. There is no path from v_7 to v_1 due to the loss of edge (v_8,v_1), which is discarded during the optimization phase since there is a shorter edge (v_8,v_7) satisfying $\angle v_7 v_8 v_1 < \frac{\pi}{3}$.

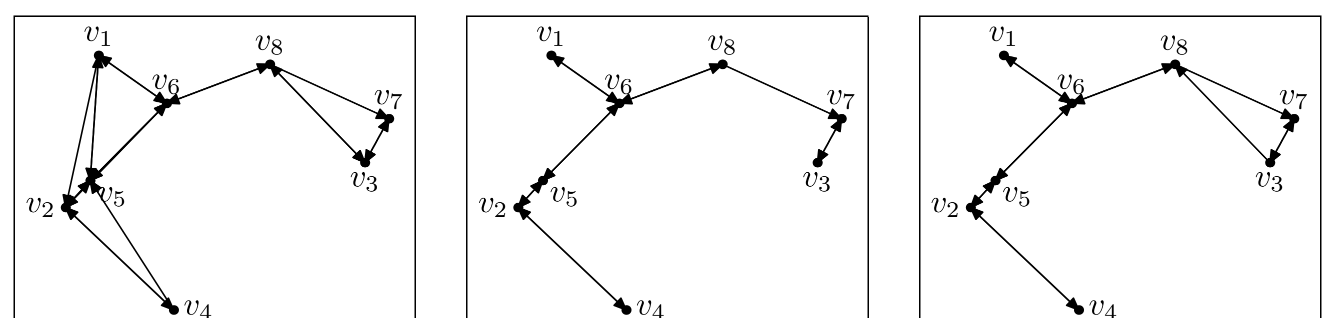


(a) Original topology (without topology control) is strongly connected.

(b) Topology by $CBTC(\frac{5}{6}\pi)$ without optimization is not strongly connected: there is no path from v_5 to v_7.

(c) Topology by DLMST is strongly connected.

Fig. 3. An example that shows $CBTC(\frac{5}{6}\pi)$ without optimization may also render disconnectivity in heterogeneous networks. There is no path from v_5 to v_7 due to the loss of edge (v_3,v_7), which is discarded by v_3 since v_2, v_4 and v_8 have already provided the necessary coverage.

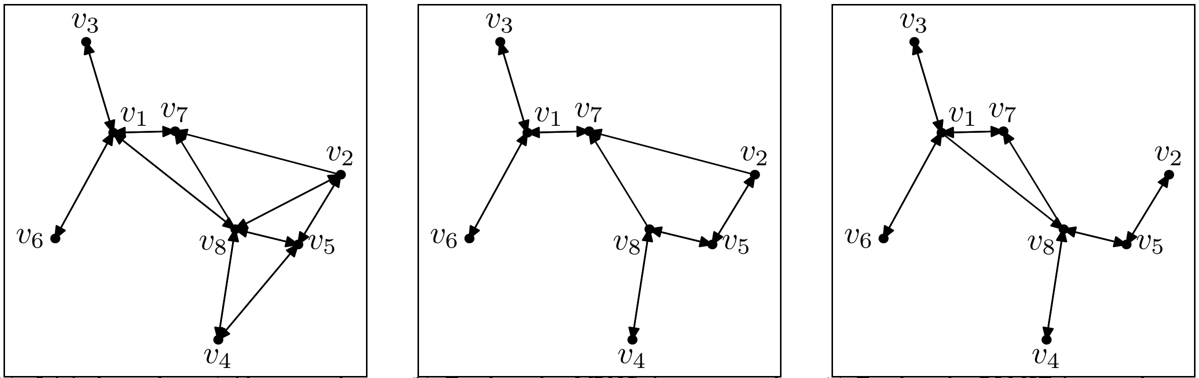


(a) Original topology (without topology control) is strongly connected.

(b) Topology by RNG is not strongly connected: there is no path from v_7 to v_8.

(c) Topology by DLMST is strongly connected.

Fig. 4. An example that shows RNG may render disconnectivity in heterogeneous networks. There is no path from v_7 to v_8 due to the loss of edge (v_3,v_8), which is discarded since $|(v_3, v_7)| < |(v_3, v_8)|$, and $|(v_8, v_7)| < |(v_3, v_8)|$.



(a) Original topology (without topology control) is strongly connected.
(b) Topology by MRNG is not strongly connected: there is no path from v_7 to v_8.
(c) Topology by DLMST is strongly connected.

Fig. 5. An example that shows MRNG may render disconnectivity in heterogeneous networks. There is no path from v_7 to v_8 due to the loss of edge (v_1,v_8), which is discarded since $|(v_1, v_7)| < |(v_1, v_8)|$, and $|(v_8, v_7)| < |(v_1, v_8)|$.

give several examples and motivate the need for new localized topology control algorithms.

We first give an example in Fig. 2 (a)-(c) that shows the optimization phase in $CBTC(\frac{5}{6}\pi)$ [6] may lead to disconnectivity (note that in Figs. 2–4 we use an arrow to represent a link from u to v). As a matter of fact, as shown in *Fig. 3* (a)-(b) the network topology derived under $CBTC(\frac{5}{6}\pi)$ without optimization may still be disconnected, when the algorithm is directly applied to a heterogeneous network.

Similarly we show in Fig. 4 (a)-(b) that the network topology derived under RNG may be disconnected when the algorithm is directly applied to a heterogeneous network. As RNG is defined for undirected graphs, one may tailor the definition of RNG for directed graphs. One natural extended definition is given below.

Definition 11 (Neighbor Relation in MRNG): For *Modified Relative Neighborhood Graph* (MRNG), $u \xrightarrow{MRNG} v$ if and only if there does not exist a third node p such that $w(u,p) < w(u,v), d(u,p) \leq r_u$ and $w(p,v) < w(u,v), d(v,p) \leq r_v$ (Fig. 1(b)).

As shown in Fig. 5 (a)-(b), the topology derived under MRNG may still be disconnected (We will give another variation of RNG for directed graphs in the next section).

IV. DRNG AND DLMST

In this section, we propose two localized topology control algorithms for heterogeneous wireless multi-hop networks with non-uniform transmission ranges: Directed Relative Neighborhood Graph (DRNG) and Directed Local Minimum Spanning Tree (DLMST). In both algorithms, the topology is derived by having each node build its neighbor set and adjust its transmission power based on locally collected information. Several nice properties of both algorithms will be discussed in Section V.

Both algorithms are composed of three phases:

1) *Information Collection*: each node collects the local information of neighbors such as position and *id*, and identifies the *Reachable Neighborhood* N^R.
2) *Topology Construction*: each node defines (in compliance with the algorithm) the proper list of neighbors for the final topology using the information in N^R.
3) *Construction of Topology with Only Bi-Directional Links* (Optional): each node adjusts its list of neighbors to make sure that all the edges are bi-directional.

a) Information collection: The information needed by each node u for topology control is the information of its reachable neighborhood N^R. This can be obtained locally, in the case of homogeneous networks, by having each node broadcast periodically a Hello message using its maximal transmission power. The information contained in a Hello message should at least include the node *id* and the position of the node. These periodic messages can be sent either in the data channel or in a separate control channel. In heterogeneous networks, having each node broadcast a Hello message using its maximal transmission power may be insufficient. For example, as shown in Fig. 6, v_1 is unable to know the position of v_4 since v_4 cannot reach v_1. We will treat this issue rigorously in Section V-D. For the time being, we assume that by the end of the first phase every node u obtains its N_u^R.

b) Topology construction: First we define the neighbor relation used in both algorithms.

Definition 12 (Neighbor Relation in DRNG): For *Directed Relative Neighborhood Graph* (DRNG), $u \xrightarrow{DRNG} v$ if and only if $d(u,v) \leq r_u$ and there does not exist a third node p such that $w(u,p) < w(u,v)$ and $w(p,v) < w(u,v), d(p,v) \leq r_p$ (see Fig. 1(c)).

Definition 13 (Neighbor Relation in DLMST): For *Directed Local Minimum Spanning Tree Graph* (DLMST), $u \xrightarrow{DLMST} v$ if and only if $(u,v) \in E(T_u)$, where T_u is the directed local MST rooted at u that spans N_u^R. That is, node v is a neighbor of node u if and only if node v is on node u's directed local MST T_u, and is "one-hop" away from node u.

In the topology construction phase of DLMST, each node u computes a directed MST that spans N_u^R and takes on-tree nodes that are one hop away as its neighbors. The algorithm to compute a directed MST was first proposed by Chu and Liu

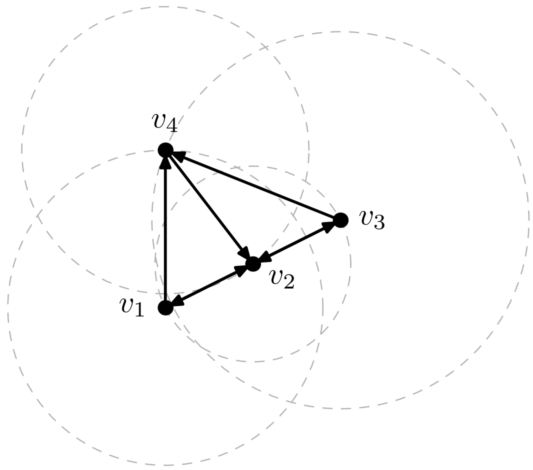


Fig. 6. An example that shows having each node broadcast a Hello message using its maximal transmission power may be insufficient for some nodes (e.g., node v_1) to know their reachable neighborhood. This figure also serves to show that given an arbitrary direct graph, it may be impossible to derive a bi-directional topology.

[18], and was reinvented by Edmonds [19] and Bock [20]. An efficient implementation was given by Tarjan [21] (see also [22]), which is $O(e \log v)$ in the worse case, $O(v \log^2 v + e)$ on average, and can be modified to be $O(v^2)$, where v is the number of nodes and e is the number of edges in G_u^R.

Each node can broadcast its own maximal transmission power in the Hello message. By measuring the receiving power of Hello messages, each node u can determine the specific power level required to reach each of its neighbors [10]. Node u then uses the power level that can reach its farthest neighbor as its transmission power. This approach can be applied to any propagation channel model.

c) Construction of topology with only bi-directional edges: As illustrated in the previous section, some links in G_{DLMST} may be uni-directional. There are two possible solutions: one can (1) enforce all the uni-directional links in G_{DLMST} to become bi-directional; or (2) delete all the uni-directional links in G_{DLMST}. We will discuss these solutions in Section V-B.

V. Properties of DRNG and DLMST

In this section, we discuss the connectivity, bi-directionality and degree bound of DLMSTLMST and DRNG. We always assume G is strongly connected, i.e., $u \Rightarrow v$ in G for any $u, v \in V(G)$.

A. Connectivity

Lemma 1: For any edge $(u, v) \in E(G) - E(G_{DLMST})$, let $P = (p_0 = u, p_1, p_2, \ldots, p_{m-1}, p_m = v)$ $((p_i, p_{i+1}) \in E(T_u), i = 0, 1, \ldots, m-1)$ be the unique path from u to v on T_u, then we have $w(p_{m-1}, v) < w(u, v)$.

Proof: We prove by contradiction. Suppose $w(p_{m-1}, v) > w(u, v)$, we can construct another directed spanning tree T'_u rooted at u with less weight, by replacing edge (p_{m-1}, v) with (u, v) and keeping all the other edges in T_u unchanged. This contradicts to the assumption that T_u is the local directed MST. ■

Lemma 2: Let T be the global directed MST of G rooted at any node $w \in V(G)$, then $E(T) \subseteq E(G_{DLMST})$.

Proof: For any edge $(u, v) \in E(T)$, we prove by contradiction. Suppose $(u, v) \notin E(G_{DLMST})$. Since v is on the directed local MST T_u, there exists a unique path $(p_0 = u, p_1, p_2, \ldots, p_{m-1}, p_m = v)$ from u to v, where $(p_i, p_{i+1}) \in E(T_u), i = 0, 1, \ldots, m-1$. We have $w(p_{m-1}, v) < w(u, v)$ by Lemma 1. By replacing edge (u, v) with (p_{m-1}, v) and keeping all the other edges in T unchanged, we can construct another global directed spanning tree T' rooted at w that has a less weight than T. This contradicts to the assumption that T is the global MST rooted at w. ■

Theorem 1 (Connectivity of DLMST): If G is strongly connected, then G_{DLMST} is also strongly connected.

Proof: For any two nodes $u, v \in V(G)$, there exists a unique global MST T rooted at u since G is strongly connected. Since $E(T) \subseteq E(G_{DLMST})$ by Lemma 2, there is a path from u to v in G_{DLMST}. ■

Lemma 3: For any edge $(u, v) \in E(G)$, we have $u \Rightarrow v$ in G_{DRNG}.

Proof: Let all the edges $(u, v) \in E(G)$ be sorted in the increasing order of $w(u, v)$, i.e., $w(u_1, v_1) < w(u_2, v_2) < \ldots < w(u_l, v_l)$, where l is the total number. We prove by induction.

1) *Basis*: The first edge (u_1, v_1) satisfies $w(u_1, v_1) = \min_{(u,v) \in E(G)}\{w(u, v)\}$. We have $d(u_1, v_1) \leq r_{min}$, otherwise G cannot be strongly connected. For any third node p, we have $w(u, p) > w(u, v)$ and $w(v, p) > w(v, u)$. By definition, $u_1 \overset{DRNG}{\longleftrightarrow} v_1$, which means $u_1 \Rightarrow v_1$ in G_{DRNG}.
2) *Induction*: Assume the hypothesis holds for all edges $(u_i, v_i), 1 \leq i < k$, we prove $u_k \Rightarrow v_k$ in G_{DRNG}. If $u_k \overset{DRNG}{\longrightarrow} v_k$, then $u_k \Rightarrow v_k$. Otherwise, there exists a third node p such that $w(u_k, p) < w(u_k, v_k), d(u_k, p) \leq r_{u_k}$ and $w(p, v_k) < w(u_k, v_k), d(p, v_k) \leq r_p$. Since (u_k, p) and (p, v_k) are edges in $E(G)$ with less weight than (u_k, v_k), we can apply the induction hypothesis to both edges. We have $u_k \Rightarrow p$,and $p \Rightarrow v_k$, thus $u_k \Rightarrow v_k$ in G_{DRNG}.

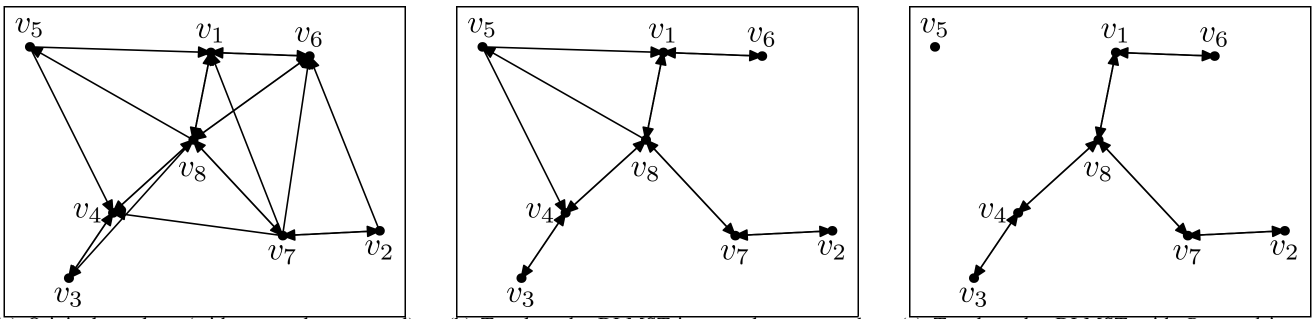
■

Theorem 2 (Connectivity of DRNG): If G is strongly connected, then G_{DRNG} is also strongly connected.

Proof: For any two nodes $u, v \in V(G)$, since G is strongly connected, there exists a path $(p_0 = u, p_1, p_2, \ldots, p_{m-1}, p_m = v)$ from u to v, such that $(p_i, p_{i+1}) \in E(G), i = 0, 1, \ldots, m-1$. Thus $p_i \Rightarrow p_{i+1}$ in G_{DRNG} by Lemma 3. Therefore, $u \Rightarrow v$ in G_{DRNG}. Hence we can conclude that G_{DRNG} is strongly connected. ■

B. Bi-directionality

Now we discuss the bi-directionality property of DRNG and DLMST. Since *Addition* may not always result in bi-directional topologies, we first apply *Removal* to topologies by

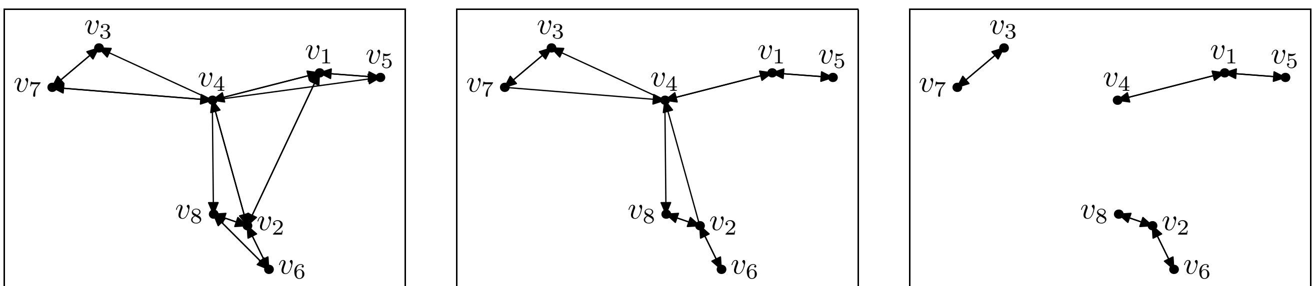


(a) Original topology (without topology control) is strongly connected.

(b) Topology by DLMST is strongly connected.

(c) Topology by DLMST with *Removal* is not strongly connected: v_5 is not connected to other nodes.

Fig. 7. An example that shows DLMST with *Removal* may result in disconnectivity.



(a) Original topology (without topology control) is strongly connected.

(b) Topology by DRNG is strongly connected.

(c) Topology by DRNG with *Removal* is not strongly connected: there are 3 components.

Fig. 8. An example that shows DRNG with *Removal* may result in disconnectivity.

DLMST and DRNG. It turns out the simple *Removal* operation may lead to disconnectivity. Examples are given in Figs. 7–8 to show, respectively, that DLMST and DRNG with *Removal* may result in disconnectivity.

In general, G may not be bi-directional if the transmission ranges are non-uniform. Since the maximal transmission range can not be increased, it may be impossible to find a bi-directional connected subgraph of G for some cases. An example is given in *Fig. 6*: v_1 can reach v_2 and v_4, v_2 can reach v_1 and v_3, v_3 can reach v_2 and v_4, and v_4 can reach v_2 only. *Addition* does not lead to bi-directionality since all edges entering or leaving v_4 are uni-directional with all nodes already transmitting with their maximal power. On the other hand, *Removal* will partition the network. In this example, although the graph G is strongly connected, its subgraph with the same vertex set cannot be both connected and bi-directional.

Now we show that bi-directionality can be ensured if the original topology is both strongly connected and bi-directional.

Lemma 4: If an edge $(u_0, v_0) \in E(G)$ satisfies $w(u_0, v_0) = \min_{(u,v)\in E(G)}\{w(u, v)\}$, then $u_0 \xleftrightarrow{DLMST} v_0$, i.e., v_0 and u_0 are neighbors of each other in G_{DLMST}.

Proof: We prove by contradiction. Assume v_0 is not a neighbor of u_0 in G_{DLMST}. We have $d(u_0, v_0) \le r_{min}$, otherwise G cannot be strongly connected. Thus $d(u_0, v_0) \le r_{u_0}, d(v_0, u_0) \le r_{v_0}$, which means $v_0 \in N^R_{u_0}$ and $u_0 \in N^R_{v_0}$. Consequently, v_0 is on the directed local MST T_{u_0} rooted at u_0. Now we find the edge $(p, v_0) \in E(T_{u_0})$ that is incident to v_0. $p \ne u_0$ by our assumption. Since $w(p, v_0) > w(u_0, v_0)$, replacing (p, v_0) with (u_0, v_0) will result in a new directed local spanning tree T'_{u_0} with a smaller cost than T_{u_0}, which is a contradiction. Therefore, $u_0 \xrightarrow{DLMST} v_0$. It can also be proved that $v_0 \xrightarrow{DLMST} u_0$ using similar arguments. Thus we have $u_0 \xleftrightarrow{DLMST} v_0$. ■

Lemma 5: If the original topology G is strongly connected and bi-directional, then any edge $(u, v) \in E(G)$ satisfies that $u \Leftrightarrow v$ in G_{DLMST}.

Proof: For all the node pairs $[u, v] : (u, v) \in E(G)$ ($(v, u) \in E(G)$ since G is bi-directional), let them be sorted in the increasing order of $w(u, v)$, i.e., $w(u_1, v_1) < w(u_2, v_2) < \ldots < w(u_l, v_l)$ where l is the total number. We prove by induction.

1) *Basis*: The first pair (u_1, v_1) satisfies $w(u_1, v_1) = \min_{(u,v)\in E(G)}\{w(u, v)\}$. Thus $u_1 \xleftrightarrow{DLMST} v_1$ by *Lemma 4*, which means $u_1 \Leftrightarrow v_1$ in G_{DLMST}.
2) *Induction*: Assume the hypothesis holds for all pairs $(u_i, v_i), i < k$, we prove $u_k \Leftrightarrow v_k$. If $u_k \xleftrightarrow{DLMST} v_k$, then $u_k \Leftrightarrow v_k$. Otherwise without loss of generality, we assume that v_k is not a neighbor of u_k's in G_{DLMST}. Thus v_k is on the directed local MST T_{u_k} and there exists a unique path $(p_0 = u_k, p_1, p_2, \ldots, p_{m-1}, p_m = v_k)$ from node u_k to node v_k, where $(p_i, p_{i+1}) \in E(T_{u_k}), i = 0, 1, \ldots, m-1$. Given that T_{u_k} is the

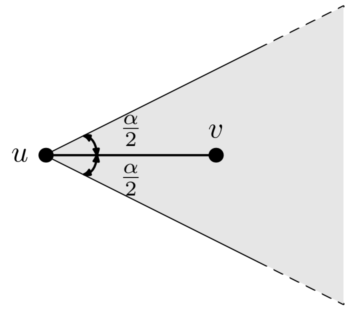


Fig. 9. The definition of $Cone(u, \alpha, v)$.

unique local MST rooted at u_k, we have $w(p_i, p_{i+1}) < w(u_k, v_k)$, since otherwise we can construct another directed spanning tree with a less weight by replacing (p_i, p_{i+1}) with (u_k, v_k), (p_j, p_{j+1}) with (p_{j+1}, p_j) for all $k \le j \le m-1$, and keeping all the other edges in T_{u_k} unchanged. Applying the induction hypothesis to each edge $(p_i, p_{i+1}), i = 0, 1, \ldots, m-1$, we have $p_i \Leftrightarrow p_{i+1}$, thus $u_k \Leftrightarrow v_k$ in G_{DLMST}. ■

Theorem 3: If the original topology G is strongly connected and bi-directional, then G_{DLMST} and G_{DRNG} are also strongly connected and bi-directional after *Addition* or *Removal*.

Proof: For any two nodes $u, v \in V(G)$, there exists at least one path $p = (w_0 = u, w_1, w_2, \cdots, w_{m-1}, w_m = v)$ from u to v, where $(w_i, w_{i+1}) \in E(G), i = 0, 1, \cdots, m-1$. Since $w_i \Leftrightarrow w_{i+1}$ in G_{DLMST} by Lemma 5, we have $u \Leftrightarrow v$ in G_{DLMST}. Also in the proof of Lemma 3, we are only able to prove $u_k \Rightarrow v_k$ because edge (v_k, u_k) may not exist. Given G is bi-directional, we should be able to prove that $u_k \Leftrightarrow v_k$. Therefore, $w_i \Leftrightarrow w_{i+1}$ in G_{DRNG}, which means $u \Leftrightarrow v$ in G_{DRNG}. The same results still hold after *Addition* or *Removal*, since all links in p are bi-directional and the removal of uni-directional links does not affect the existence of such a path. ■

C. Degree Bound

It has been observed that any minimum spanning tree of a simple undirected graph in the plane has a maximum node degree of 6 [23]. However, this bound does not hold for directed graphs. An example is shown in Fig. 10, where node u has 18 neighbors. In this section, we will discuss the node degree in the topology by DLMST and DRNG.

Definition 14 (Disk): $Disk(u, r)$ is the disk centered at node u with a radius of r.

Definition 15 (Cone): $Cone(u, \alpha, v)$ is the unbounded shaded region shown in Fig. 9.

Lemma 6: Given three nodes $u, v, w \in V(G_{DLMST})$ satisfying $w(u, v) > w(u, w)$ and $w(u, v) > w(w, v)$, $d(w, v) \le r_w$, then $u \nrightarrow v$ in G_{DLMST}.

Proof: We only need to consider the case when $d(u, v) \le r_u$ since $d(u, v) > r_u$ would imply $u \nrightarrow v$. Assume $u \rightarrow v$. Since $d(u, w) \le d(u, v) \le r_u$, there exists a unique path

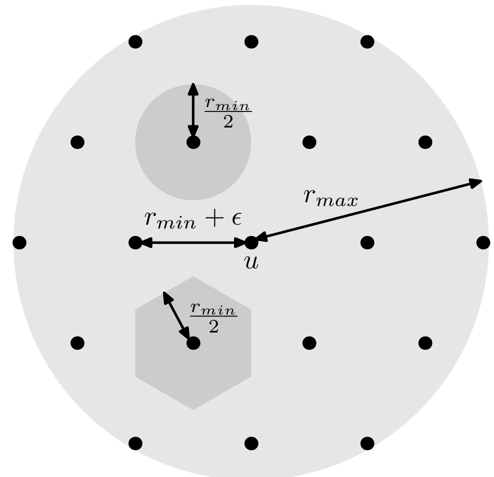


Fig. 10. An example that shows the out degree in a heterogenous network can be very large. The transmission range of u is r_{max} and the transmission range for all other nodes is r_{min}, where $r_{max} = 2(r_{min} + \epsilon)$, $\epsilon > 0$. All nodes are so arranged that the distance between any node and its closest neighbor is $r_{min} + \epsilon$. Therefore, the only links in the network are those from u to all the other nodes. Since relaying packets is impossible, u has to use its maximal transmission power and keeps all 18 neighbors.

$p = (v_0 = u, v_1, v_2, \cdots, v_{m-1}, v_m = w)$ on T_u from node u to node w, where $(v_i, v_{i+1}) \in E(T_u), i = 0, 1, \cdots, m-1$. If v is on the path p, replacing edge (u, v) with edge (u, w) and keeping all other edges unchanged in T_u will result in a spanning tree of G_u with a smaller weight. If v is not on p, replacing edge (u, v) with edge (w, v) and keeping all other edges unchanged in T_u will result in a spanning tree of G_u with a smaller weight. Both scenarios contradict with the fact that T_u is the unique minimum spanning tree of G_u. ■

Corollary 1: If v is a neighbor of u's in G_{DLMST}, and $d(u, v) \ge r_{min}$, then u can not be have any other neighbor inside $Disk(v, r_{min})$.

Theorem 4: For any node $u \in V(G_{DLMST})$, the number of neighbors in G_{DLMST} that are inside $Disk(u, r_{min})$ is at most 6.

Proof: Let $N(u)$ be the set of neighbors of u in G_{DLMST} that are inside $Disk(u, r_{min})$. Let the nodes in $N(u)$ be ordered such that for the ith node w_i and the jth node w_j $(j > i)$, $w(u, w_j) > w(u, w_i)$. By Lemma 6, we have $w(u, w_j) \le w(w_i, w_j)$ (otherwise $u \nrightarrow w_j$). Thus $\angle w_i u w_j \ge \pi/3$, i.e., node w_j cannot reside inside $Cone(u, 2\pi/3, w_i)$. Therefore, node u cannot have neighbors other than node w_i inside $Cone(u, 2\pi/3, w_i)$. By induction on the rank of nodes in $N(u)$, the maximal number of neighbors that u can have is at most 6. ■

Theorem 5: The out degree of node in G_{DLMST} is bounded by a constant that depends only on r_{max} and r_{min}.

Proof: For any node u in G_{DLMST}, there are at most 6 neighbors inside $Disk(u, r_{min})$ from Theorem 4. Also from Corollary 1, the set of disks $\{Disk(v, \frac{r_{min}}{2}) : v \in N_{DLMST}(u), v \notin Disk(u, r_{min})\}$ are disjoint. Therefore, the

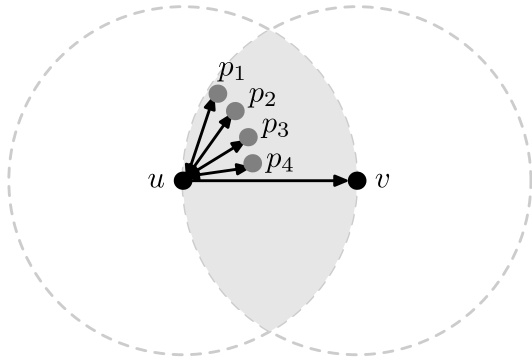


Fig. 11. The out degree may be unbounded in G_{DRNG}.

total number of neighbors of u is bounded by:

$$c_1 = 6 + \left\lceil \frac{\pi[(r_{max} + \frac{r_{min}}{2})^2 - (\frac{r_{min}}{2})^2]}{\pi(\frac{r_{min}}{2})^2} \right\rceil = 4\lceil \beta(\beta+1) \rceil + 6,$$

where $\beta = \frac{r_{max}}{r_{min}}$. Actually we can observe that Fig. 10 shows the scenario where the maximum out degree of u is achieved if $\epsilon \to 0$. Therefore, we can further tighten the bound. Since the hexagonal area (as shown in Fig. 10) centered at every neighbor of u is disjoint with each other, the total number of neighbors of u is bounded by:

$$c_2 = \left\lceil \frac{\pi(r_{max} + \frac{r_{min}}{\sqrt{3}})^2}{\frac{\sqrt{3}}{2} r_{min}^2} \right\rceil - 1 = \left\lceil \frac{2\pi}{\sqrt{3}}(\beta + \frac{1}{\sqrt{3}})^2 \right\rceil - 1.$$

■

The bound given in Theorem 4 is actually quite large. We will show in Section VI that the average maximum degree is much smaller for networks with random distributed nodes. Also note that what has been discussed so far is actually the *logical* node degree, i.e., the number of neighbors. In practice, it is more important to consider the *physical* node degree, i.e., the number of nodes within the transmission radius. If omni-directional antennas are used, the physical degree cannot be bounded for an arbitrary topology. However, with the help of directional antennas, we will be able to bound the physical degree given that the logical degree is bounded under DLMST (except in some extreme cases, e.g., a large number of nodes are of the same distance from one node). The idea is that, when transmitting to a specific neighbor, node u should adjust the direction and limit the transmission power so that no other nodes will be affected.

Notice that the out degree is not bounded in G_{DRNG}. An example is given in Fig. 11. For all p_i that lies inside the shaded area, as long as $r_{p_i} < d(p_i, v)$, the edge (u, v) in G_{DRNG} will not exclude edges $(u, p_i), i = 1, 2, \ldots$. As a result, the out degree of u is unbounded.

D. Localized Algorithms

As mentioned in Section IV, in the case that nodes may have different maximal transmission powers, the operation of having each node u broadcast its own position information to all the other nodes within r_u is not sufficient to ensure each node u obtains the information of reachable neighborhood N_u^R (Fig. (6)). Fortunately with the desirable properties of DRNG and DLMST proved in Sections V-A and V-B, we show that it is sufficient for node u to collect neighborhood information only from nodes whose maximal transmission range covers node u. That is, the original information exchange algorithm that requires only "one-hop" information suffices.

Consider a directed simple graph with less edges: $G' = (V(G'), E(G'))$, where $E(G') = \{(u, v) : d(u, v) \leq \min(r_u, r_v), u, v \in V(G)\}$. For any edge $(u, v) \in E(G')$, since $d(u, v) \leq \min(r_u, r_v)$, we have $(v, u) \in E(G')$, which means G' is bi-directional. Define $N_u^{R'} = \{v \in V(G) : d(u, v) \leq \min(r_u, r_v)\}$, $r_u{}' = \max_{v \in N_u^{R'}}\{d(u, v)\}$, where $r_u{}' \leq r_u$ since for any $v \in N_u^{R'}$, $d(u, v) \leq r_u$. Let $r_{min}{}' = \min_{v \in V}\{r_v{}'\}$ and $r_{max}{}' = \max_{v \in V}\{r_v{}'\}$. By requiring each node u to broadcast its position and id to all other nodes within r_u, we are able to determine $N_u^{R'}$ and $r_u{}'$. We can then apply DRNG and DLMST on top of G' and prove that Theorems 1-4 still hold even if the original topology is G'.

Theorem 6: Theorems 1–5 still holds if the original topology is G'.

Proof: We replace G, r_u, N_u^R, r_{min}, and r_{max} with G', $r_u{}'$, $N_u^{R'}$, $r_{min}{}'$ and $r_{max}{}'$ in the proof of Lemma 1–6 and Theorem 1–5. Then following the same line of arguments, we can prove that they still hold if the original topology is G'. ■

Theorem 7: If the original topology is G' (which is a subgraph of G), G_{DLMST} and G_{DRNG} are bi-directional after *Addition* or *Removal*.

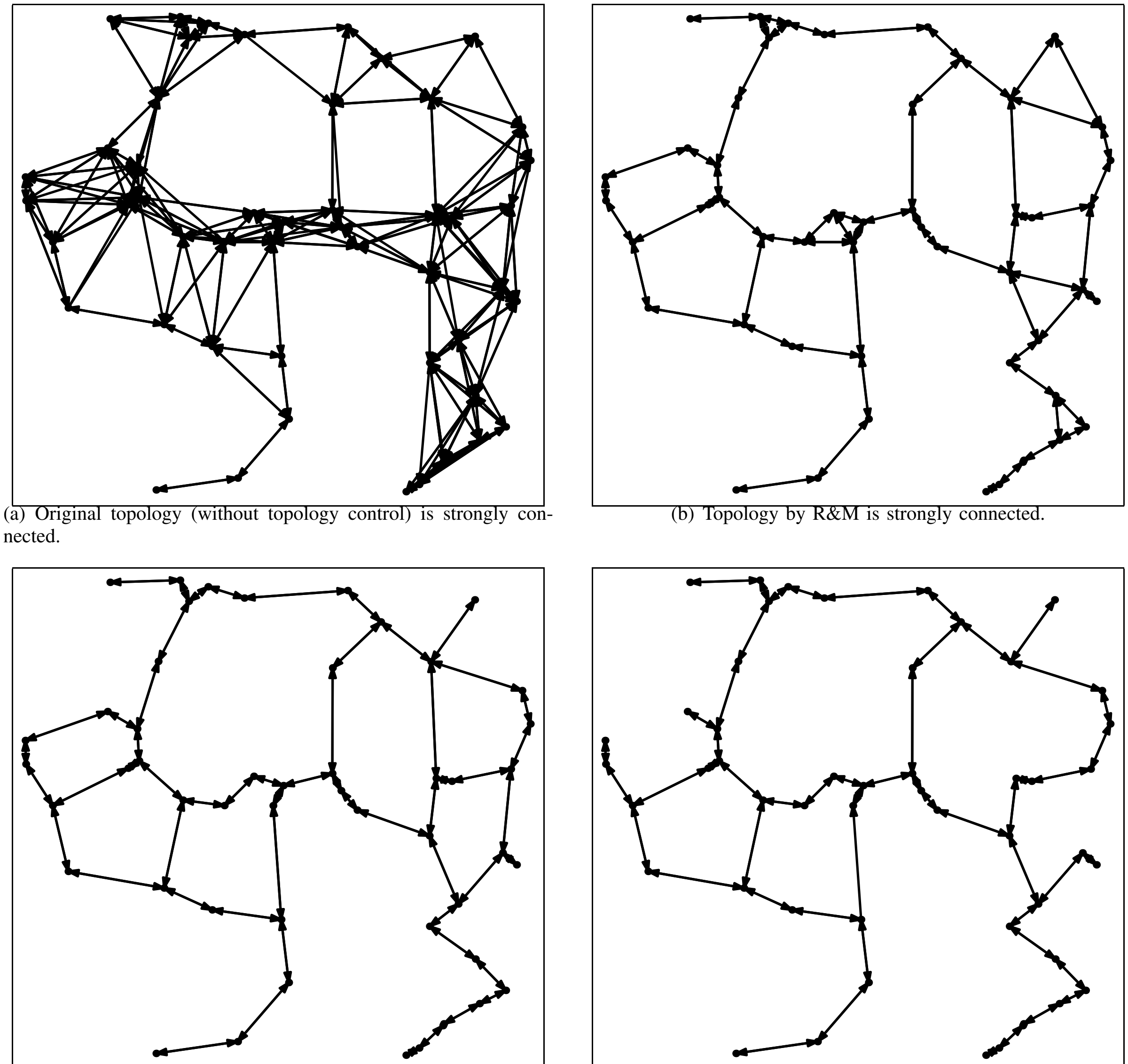
Proof: We apply Theorem 3 to G', as G' is bi-directional. ■

VI. Simulation Study

In this section, we evaluate the performance of R&M, DRNG, and DLMST by simulations. All three algorithms are known to preserve network connectivity in heterogeneous networks.

In the first simulation, 50 nodes are uniformly distributed in a $1000m \times 1000m$ region. The transmission ranges for nodes are uniformly distributed in $[200m, 250m]$. Fig. 12 gives the topologies derived using the maximal transmission power (labeled as NONE), R&M (under the two-ray ground model), DRNG, and DLMST for one simulation instance. As shown in Fig. 12, R&M, DRNG and LMST all significantly reduce the average node degree, while maintaining network connectivity. Moreover, both DRNG and DLMST outperforms R&M in the sense that fewer edges are formed in the topology.

In the second simulation, we vary the number of nodes in the region from 80 to 300, and each data point is an average of 100 simulation runs. The transmission ranges of nodes are uniformly distributed in $[10m, 250m]$. Fig. 13 shows the average radius and the average edge length for the topologies derived under NONE(no topology control), R&M, DRNG, and DLMST. DLMST outperforms the others, which implies that DLMST can provide a better spatial reuse and use less energy to communicate.



(a) Original topology (without topology control) is strongly connected.

(b) Topology by R&M is strongly connected.

(c) Topology by DRNG is strongly connected.

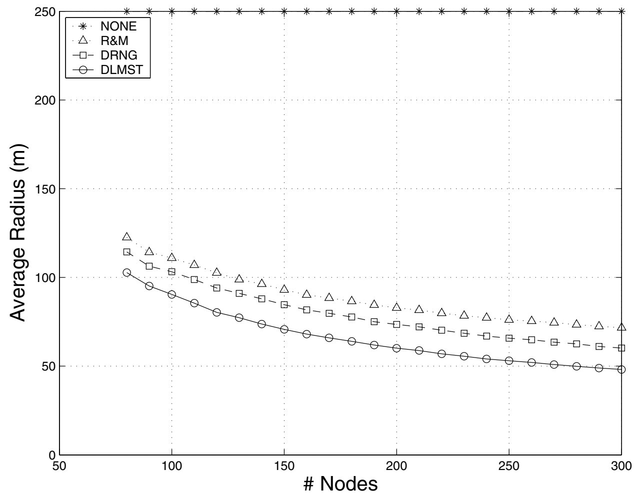
(d) Topology by DLMST is strongly connected.

Fig. 12. Topologies derived by R&M, DRNG, and DLMST.

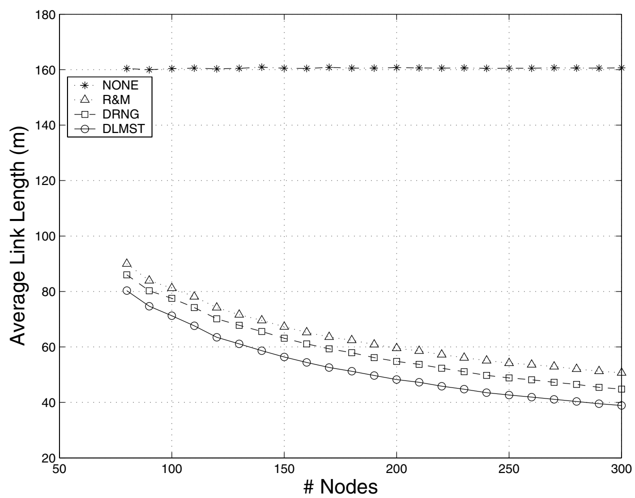
We also compare the out degree of the topologies by different algorithms. The result of NONE is not shown because the out degrees increase almost linearly with the number of nodes and are significantly larger than those under R&M, DRNG, and DLMST. Fig. 14 shows the average logical/physical out degree for the topologies derived by R&M, DRNG, and DLMST. The average out degrees under R&M and DRNG increase with the increase in the number of nodes, while those under DLMST actually decrease. Fig .15 shows the average maximum logical degree and the largest maximum logical out degree for each number of nodes. The largest maximum logical degree under DLMST is at most 4, and is well below the theoretical upper bound obtained in Theorem 5. Also DLMST has much smaller degrees than the other topologies. Similar results can be observed in Fig. 16 for physical degrees. The only difference is that the physical degrees are in general much higher than the logical degrees for the same network.

VII. Conclusions

In this paper, we have proposed two local topology control algorithms, Directed Relative Neighborhood Graph (DRNG) and Directed Local Minimum Spanning Tree (DLMST), for heterogeneous wireless multi-hop networks in which each node may have different maximal transmission ranges. We show that as most existing topology control algorithms (except R&M [4]) do not consider the fact that nodes may have different maximal transmission ranges, they render disconnected network topology when directly applied to heterogeneous networks. Then we devise DRNG and DLMST and prove that (i) both DRNG and DLMST preserve network connectivity; (ii) both DRNG and DLMST preserve network bi-directionality if *Addition* and *Remove* operations are applied to the topologies

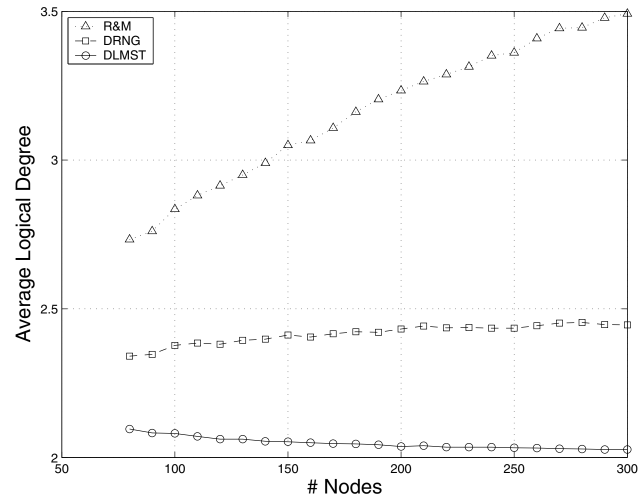


(a) Average radius.

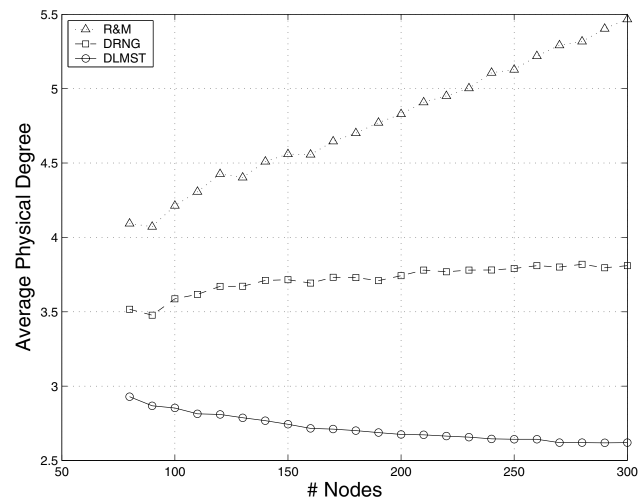


(b) Average link length

Fig. 13. Comparison of DLMST, DRNG and R&M with respect to average radius and average edge length.



(a) Average logical out degree.



(b) Average physical out degree.

Fig. 14. Comparison of R&M, DRNG and DLMST with respect to average out degree.

derived under these algorithms; and (iii) the out degree of any node is bounded in the topology derived under DLMST, while that may be unbounded under DRNG. The simulation study validates the superiority of DRNG and DLMST over R&M.

As part of our future research, we will pursue the following open problems: (1) given a topology in which each node transmits with different maximal transmission power, what is the probability that the topology is bi-directional with respect to the distribution and the density of nodes, and the distribution of the transmission ranges? and (2) How will MAC-level interference affect network connectivity and bi-directionality?

REFERENCES

[1] C. E. Jones, K. M. Sivalingam, P. Agrawal, and J. C. Chen, "A survey of energy efficient network protocols for wireless networks," *Wireless Networks*, vol. 7, no. 4, pp. 343–358, Aug. 2001.

[2] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Inform. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[3] S. Narayanaswamy, V. Kawadia, R. S. Sreenivas, and P. R. Kumar, "Power control in ad-hoc networks: Theory, architecture, algorithm and implementation of the COMPOW protocol," in *Proc. of European Wireless 2002, Next Generation Wireless Networks: Technologies, Protocols, Services and Applications*, Florence, Italy, Feb. 2002, pp. 156–162.

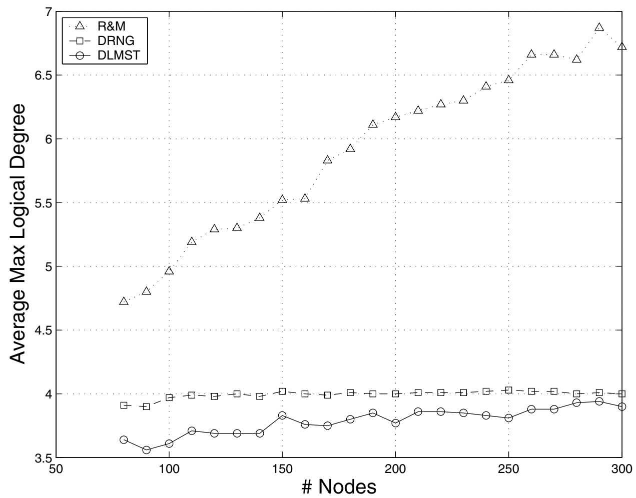
[4] V. Rodoplu and T. H. Meng, "Minimum energy mobile wireless networks," *IEEE J. Select. Areas Commun.*, vol. 17, no. 8, pp. 1333–1344, Aug. 1999.

[5] R. Ramanathan and R. Rosales-Hain, "Topology control of multihop wireless networks using transmit power adjustment," in *Proc. IEEE INFOCOM 2000*, Tel Aviv, Israel, Mar. 2000, pp. 404–413.

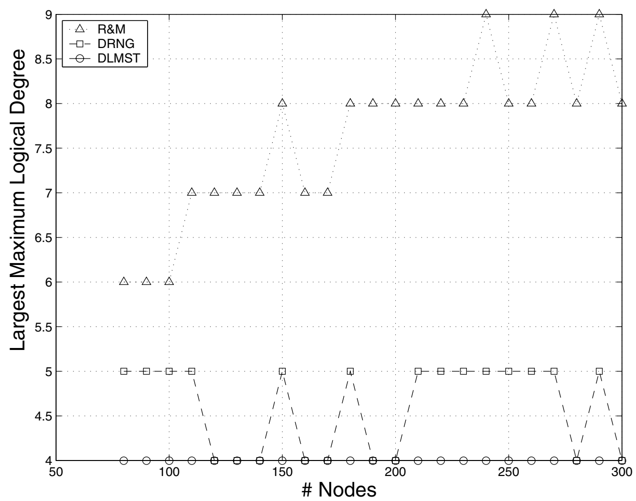
[6] L. Li, J. Y. Halpern, P. Bahl, Y.-M. Wang, and R. Wattenhofer, "Analysis of a cone-based distributed topology control algorithm for wireless multi-hop networks," in *Proc. ACM Symposium on Principles of Distributed Computing*, Newport, Rhode Island, US, Aug. 2001, pp. 264–273.

[7] V. Kawadia and P. Kumar, "Power control and clustering in ad hoc networks," in *Proc. IEEE INFOCOM 2003*, San Francisco, California, US, Apr. 2003.

[8] S. A. Borbash and E. H. Jennings, "Distributed topology control algorithm for multihop wireless networks," in *Proc. 2002 World Congress on Computational Intelligence (WCCI 2002)*, Honolulu, Hawaii, US, May 2002.

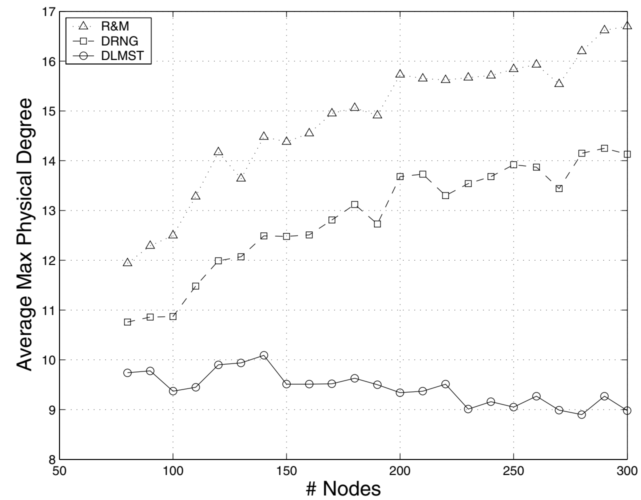


(a) Average maximum logical out degree.

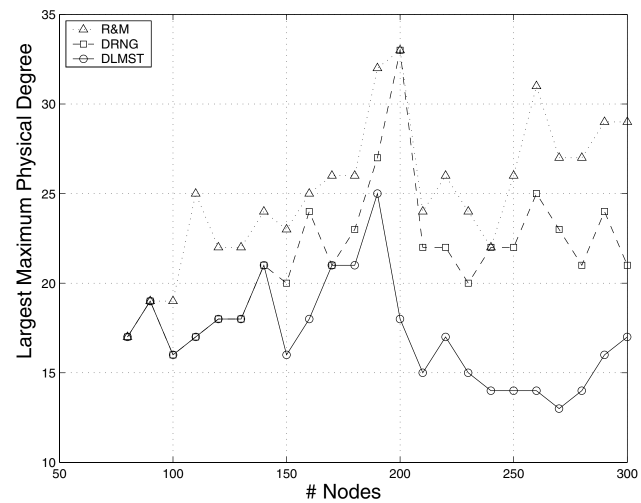


(b) Largest maximum logical out degree.

Fig. 15. Comparison of R&M, DRNG and DLMST with respect to the maximum logical degree.



(a) Average maximum physical out degree.



(b) Largest maximum physical out degree.

Fig. 16. Comparison of R&M, DRNG and DLMST with respect to the maximum physical degree.

[9] X.-Y. Li, G. Calinescu, and P.-J. Wan, "Distributed construction of planar spanner and routing for ad hoc networks," in *Proc. IEEE INFOCOM 2002*, New York, New York, US, June 2002.

[10] N. Li, J. C. Hou, and L. Sha, "Design and analysis of an MST-based topology control algorithm," in *Proc. IEEE INFOCOM 2003*, San Francisco, California, US, Apr. 2003.

[11] C. Kee, H. Jun, D. Yun, B. Kim, Y. Kim, B. W. Parkinson, T. Langenstein, S. Pullen, and J. Lee, "Development of indoor navigation system using asynchronous pseudolites," in *Proc. 13th International Technical Meeting of the Satellite Division of the Institute of Navigation (ION GPS 2000)*, Salt Lake City, Utah, US, Sept. 2000, pp. 1038–1045.

[12] T. He, C. Huang, B. M. Blum, J. A. Stankovic, and T. Abdelzaher, "Range-free localization schemes for large scale sensor networks," in *Proc. ACM MOBICOM 2003*, San Diego, California, United States, Sept. 2003, pp. 81–95.

[13] G. Toussaint, "The relative neighborhood graph of a finite planar set," *Pattern Recognition*, vol. 12, pp. 261–268, 1980.

[14] K. J. Supowit, "The relative neighborhood graph, with an application to minimum spanning trees," *Journal of the ACM*, vol. 30, no. 3, pp. 428–448, July 1983.

[15] P. Santi, D. M. Blough, and F. Vainstein, "A probabilistic analysis for the range assignment problem in ad hoc networks," in *Proc. ACM Symposium on Mobile Ad Hoc Networking and Computing (MOBIHOC) 2001*, Long Beach, California, US, Aug. 2000, pp. 212–220.

[16] S. Basagni, D. Turgut, and S. K. Das, "Mobility-adaptive protocols for managing large ad hoc networks," in *Proc. IEEE International Conference on Communications (ICC) 2002*, Helsinki, Finland, June 2001, pp. 1539–1543.

[17] M. G. J. Wu and I. Stojmenovic, "On calculating power-aware connected dominating sets for efficient routing in ad hoc wireless networks," in *Proc. International Conference on Parallel Processing*, Vancouver, British Columbia, Canada, Aug. 2002, pp. 346–354.

[18] Y. J. Chu and T. H. Liu, "On the shortest arborescence of a directed graph," *Science Sinica*, vol. 14, pp. 1396–1400, 1965.

[19] J. Edmonds, "Optimum branchings," *Journal of Research of the National Bureau of Standards*, vol. 71B, pp. 233–240, 1967.

[20] F. Bock, "An algorithm to construct a minimum spanning tree in a directed network," in *Developments in Operations Research*. New York, New York, US: Gordon and Breach Science Publishers, 1971, pp. 29–44.

[21] R. E. Tarjan, "Finding optimum branchings," *Networks*, vol. 7, pp. 25–35, 1977.

[22] P. Camerini, L. Fratta, and F. Maffioli, "A note on finding optimum branchings," *Networks*, vol. 9, pp. 309–312, 1979.

[23] C. Monma and S. Suri, "Transitions in geometric minimum spanning trees," in *Proc. ACM Symposium on Computational Geometry*, North Conway, New Hampshire, United States, 1991, pp. 239–249.