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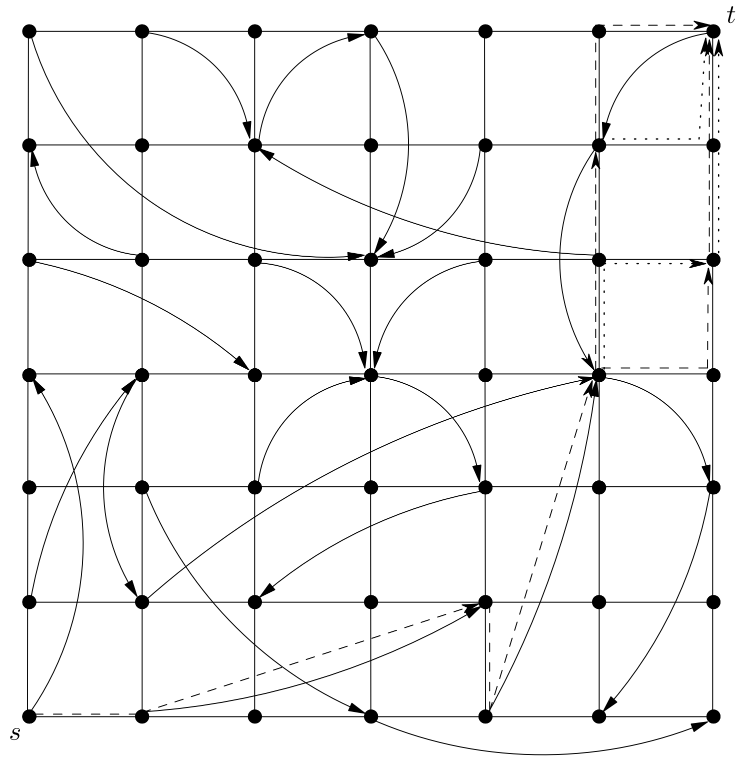
Principles of Distributed Computing
Exercise 4: Sample Solution

For questions and/or comments, you can write e-mails in English to `davide.bilo@di.univaq.it`.

1 Greedy algorithm with lookahead

a) The algorithm `Look`(2) computes one of the following four $s-t$ paths



b) The solution follows the same line of the proof of Theorem 2 of Kleinberg's paper. We start by writing a general formula which will be useful for solving also part (c) of this exercise. Let $S^k(u)$ be the set of points within lattice distance $k-1$ from u. Let B_j be the set of nodes within lattice distance 2^j of t. Without loss of generality, we can assume that $B_j \cap S^k(u) = \emptyset$ (otherwise we will change phase in 1 step). Hence we have

$$\begin{aligned}
\Pr[\exists \text{ long-range contact from } S^k(u) \text{ to } B_j] = \\
&= 1 - \Pr[\nexists \text{ any long-range contact from } S^k(u) \text{ to } B_j] \\
&= 1 - \prod_{x \in S^k(u)} \Pr[x \text{ has no long-range contact to a node in } B_j] \\
&= 1 - \prod_{x \in S^k(u)} (1 - \Pr[x \text{ has a long-range contact to a node in } B_j]) \\
&\geq 1 - \left(1 - \frac{2^{2j-1}}{4\ln(6n)2^{2j+4}}\right)^k = 1 - \left(1 - \frac{1}{128\ln(6n)}\right)^k
\end{aligned}$$

where the last inequality holds because $S^k(u)$ has at least k points within lattice distance 2^{j+2} from B_j.

For $k = 2$ we have that the probability that exists a long-range contact from $S^2(u)$ to B_j is at least $1-\left(1 - \frac{1}{128\ln(6n)}\right)^2 \geq 1-\left(1 - \frac{c}{128\ln(6n)}\right) = \frac{c}{128\ln(6n)}$ for any constant $1 \leq c \leq 255/128$, and $n \geq 1$.

Let X_j denote the total number of steps spent in phase $j, \log(\log n) \leq j < \log n$. We have

$$EX_j = \sum_{i=1}^{\infty} \Pr[X_j \geq i] \leq \sum_{i=1}^{\infty} \left(1 - \frac{c}{128\ln(6n)}\right)^{i-1} = \frac{128}{c}\ln(6n).$$

Hence, in expectation, the total number of steps spent in phase j decreases by a constant factor if compared to the total number of steps spent in phase j by algorithm `Greedy`.

So, if X denotes the total number of steps spent by the algorithm, we have $X = \sum_{j=0}^{\log n} X_j$, and so by linearity of expectation we have $EX \leq (1 + \log n)(\frac{128}{c}\ln(6n)) \leq \alpha_2(\log n)^2$ for a suitable choice of α_2.

c) For $k = \log n$ (for simplicity, assume that $\log n$ is an integer), the probability that there is a long-range contact from $S^{\log n}(u)$ to B_j is at least

$$1-\left(1 - \frac{1}{128\ln(6n)}\right)^{\log n} = 1-\left(1 - \frac{\log e}{128\log(6n)}\right)^{\log n} \geq 1-\left(1 - \frac{1}{768\log n}\right)^{\log n} \geq 1-\frac{1}{e^{\frac{1}{768}}}.$$

Let X_j denote the total number of steps spent in phase $j, \log(\log n) \leq j < \log n$. We have

$$EX_j = \sum_{i=1}^{\infty} \Pr[X_j \geq i] \leq \sum_{i=1}^{\infty} \left(1 - 1 + \frac{1}{e^{\frac{1}{768}}}\right)^{i-1} = \sum_{i=1}^{\infty} \left(\frac{1}{e^{\frac{1}{768}}}\right)^{i-1} = \frac{e^{\frac{1}{768}}}{e^{\frac{1}{768}} - 1}.$$

Hence, in expectation, the total number of steps spent in phase j decreases by a factor of $\log n$ if compared to the total number of steps spent in phase j by algorithm `Greedy`.

So, if X denotes the total number of steps spent by the algorithm, we have $X = \sum_{j=0}^{\log n} X_j$, and so by linearity of expectation we have

$$EX \leq (1 + \log n)\frac{e^{\frac{1}{768}}}{e^{\frac{1}{768}} - 1} \leq \alpha_2(\log n)$$

for a suitable choice of α_2.

2 3-dimensional Small-World networks

a) Any node u is connected to 6 nearest neighbors in the lattice (3 or 4 in the case of nodes on the boundary). Hence we have

$$\sum_{v \neq u} d(u,v)^{-3} \leq \sum_{j=1}^{3n-3} (4j^2+2)j^{-3} \leq 6 \sum_{j=1}^{3n-3} j^{-1} \leq 6 + 6\ln(3n-3) \leq 6\ln(9n)$$

where the first inequality holds because the number of nodes at distance j from u is at most

$$2 + 4j + 2\sum_{i=1}^{j} 4i = 2 - 4j + 8\sum_{i=1}^{j} i = 2 - 4j + 8\frac{j(j+1)}{2} = 4j^2 + 2.$$

b) The set B_j contains at least

$$\sum_{i=1}^{2^j}\sum_{k=1}^{i} k = 1 + \sum_{i=2}^{2^j} \frac{i(i+1)}{2} \geq 1 + \frac{1}{2}\sum_{i=2}^{2^j} i^2 = 1 + \frac{1}{2} \cdot \frac{2^j(2^j+1)(2^{j+1}+1)}{6} - \frac{1}{2} \geq \frac{2^{3j-1}}{3}$$

nodes within lattice distance 2^{j+2} of u. Hence, the message enters B_j with probability at least

$$\frac{2^{3j-1}}{18\ln(9n)2^{3j+6}} = \frac{1}{2304\ln(9n)}.$$

Let X_j be the total number of steps spent in phase $j, \log(\log) \leq i < \log n$. We have

$$EX_j = \sum_{i=1}^{\infty} \Pr[X_j \geq i] \leq \sum_{i=1}^{\infty} \left(1 - \frac{1}{2304\ln(9n)}\right)^{i-1} = 2304\ln(9n).$$

The remaining part of the exercise proceeds along the same line of the proof of Theorem 2 of Kleinberg's paper.