Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

SS 2006 Prof. R. Wattenhofer / Prof. P. Widmayer / T. Locher / R. Flury

Principles of Distributed Computing
Exercise 4: Sample Solution

1 Bad Queues in a Mesh

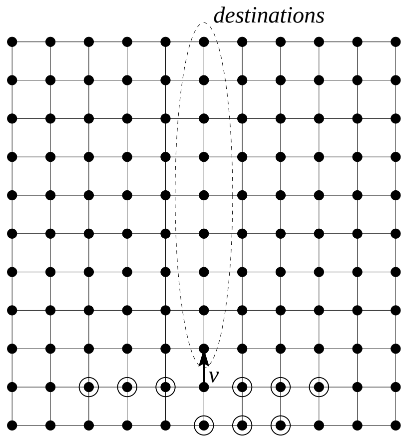

Figure 1: $m-2$ packets congest at node v

In order to obtain big queues at a node v, packets need to arrive from all three possible directions in each step. Therefore, the maximum number of destinations from one direction in a column is $m-2$. See Figure 1. In each step, the queue grows by 2 and there are $(m-2)/3$ steps. Thus the queue size grows to

$$\frac{2}{3}(m-2)$$

2 Good Queues in a Mesh

Following the lead given in the exercise we want to bound the probability P_{2em} that a particular column contains $2em$ or more destination packets. Analogous to the proof of Theorem 4.10 in the lecture, we have

$$P_{2em} < \binom{m^2}{2em} \cdot \left(\frac{1}{m}\right)^{2em} \tag{1}$$

(since we put $2em$ out of the m^2 destination packets in that column, each with a probability $1/m$). Using the inequality of the lecture (in the same proof) we can further simplify this to

$$P_{2em} < \left(\frac{em^2}{2em}\right)^{2em} \left(\frac{1}{m}\right)^{2em} = \left(\frac{1}{2}\right)^{2em} \tag{2}$$

to obtain that the probability for a single column to contain more than $2em$ packets is "really small" (i.e. in $o(2^{-m})$).

Since we want a bound on the column with the maximum number of destination packets, we can compute the probability P_{2em}^{all} that all m columns contain $2em$ or more packets:

$$P_{2em}^{\text{all}} \leq \sum_{i=1}^{m} P_{2em} = mP_{2em} \tag{3}$$

since the probability for all columns is the union of the probabilities that in each column there are more than $2em$ packets. The union of probabilities is upper bounded by their sum. Plugging (2) into (3) we get that

$$P_{2em}^{\text{all}} < \frac{m}{2^{2em}} < \frac{1}{m^2} \tag{4}$$

where we used that $m/2^m \leq 1/m^2$ for large m since an exponential function grows faster than any polynomial.

Altogether, the argument is then as follows: The probability that all columns contain less than $O(m)$ packets is high, namely in $1 - O(1/m^2)$. Therefore, we also have a high probability that the column containing the most number of destinations also gets only $O(m)$ packets. To route a packet along a row takes at most $m-1$ time steps. Once it has arrived at the designated column, it will have to wait for at most $O(m)$ other packets (with high probability). Altogether each packet then needs time $O(m)$ to arrive at its destination.