Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Computer Engineering and Networks Laboratory
TEC, NSG, DISCO

HS 2017

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Discrete Event Systems

Exam Questions 1 - 4

Saturday, 3rd February 2018, 09:00–11:00.

Do not open until told to by the supervisors!

The exam lasts 120 minutes, and there is a total of 120 points. The maximal number of points for each question is indicated in parentheses. Your answers must be in English. Be sure to always justify your answers.

Give your solutions on the page corresponding to the exam question and/or the empty one(s) following it. In case you run out of space, we also added extra pages – in case you use them, please indicate which question you are solving! Should even this not be enough, please contact a supervisor.

Please write down your name and Legi number (student ID) in the boxes below. Once the exam starts, also write your name on every page in the top right corner.

Name	**Legi-Nr.**

Points

Question	Topic	Achieved Points	Maximal Points
1	Regular Languages		25
2	Context-free Languages		15
3	King of the Hill		15
4	GPU Rental		25

1 Regular Languages (25 points)

1.1 NFA, DFA, REX [15 points]

Consider the language $L = \{w \mid w \in \{0,1\}^*$ such that w contains an even number of zeros or exactly two ones$\}$. For example, the strings 00011 and 111 belong to L, while the strings 100000 and 1110 do not. The empty string belongs to L.

a) [5] Draw a NFA that recognizes L with 6 or less states.

b) [6] Draw a DFA that recognizes L with 8 or less states.

c) [4] Give a Regular Expression for L.

Additional space for 1.1

1.2 Regular or not? [10 points]

Are the following languages regular? If so, exhibit a finite automaton (deterministic or not) or a regular expression for it. If not, prove it formally using the pumping lemma or the closure properties of regular languages.

a) [5] $L = \{0^n \mid n \text{ is a power of } 2\}$.

b) [5] $L = \{w \mid w \in \{0,1\}^* \wedge \text{ there is no } x \text{ such that } w = xx\}$.

Additional space for 1.2

2 Context-free languages (15 points)

2.1 Write me a grammar [5 points]

Give a context-free grammar (the production rules) for the following language:

$$L = \{a^n b^m \mid 0 \leqslant n \leqslant m \leqslant 3n\}$$

2.2 Ambiguous or Not? `[5 points]`

Let G be the context-free grammar given by the following production rules:

$$
\begin{aligned}
S &\to 0A \mid 1B \\
A &\to 0AA \mid 1S \mid 1 \\
B &\to 1BB \mid 0S \mid 0
\end{aligned}
$$

Is G ambiguous? Explain your answer.

2.3 Draw me a PDA [5 points]

Consider the following Language:

$$L = \{a^{2n}b^{3n} \mid n \geqslant 0\}$$

Design a PDA that accepts L. The designed PDA must have *9 or less* states.

Additional space for question 2

3 King of the Hill (15 points)

King of the Hill is a game played between two players P1 and P2. The playground is a terrain with seven hills. There is a cage on top of some hill. After some random time (Poisson process with expected time 1 minute) the cage will disappear on the hill, and immediately appear again on another hill. The objective of the players is to be in the cage alone. Whenever a player is alone in the cage for time t, the player will earn t points. The time taken by P1 and P2 to get to the cage is a Poisson process with expected value 1/2 minute and 1 minute, respectively. When both players are inside the cage, no player will earn points, and there is a shootout between them. The shootout duration is a Poisson process and takes 1/2 minute on average. P1 wins the shootout with probability p and P2 with probability $1 - p$. The player who wins the shootout remains in the cage, whereas the other player again spawns somewhere in the playground, heading again for the cage.

a) [5] Model the game as a Continuous Time Markov Chain.

b) [5] If the game is played long enough, what is the probability that the cage is empty?

c) [5] Assume P2 wins each and every shootout, i.e., $p = 0$. If the game is played long enough, which player gains more points in expectation?

Additional space for question 3

4 GPU (25 points)

Alice needs a graphics processing unit (GPU) to do deep learning research. She can

- buy a GPU at time t for the price $b(t)$,
- or rent a GPU with rental cost rate $r(t)$: If Alice rents a GPU from time t_1 to t_2, then the total rental cost is $\int_{t_1}^{t_2} r(t)dt$.

Alice does not know the time (denoted by T) when she will not be interested in deep learning anymore. That is, after unknown time T, she does not need a GPU. Let's assume that

- $b : R_+ \mapsto R_+$ is a non-increasing function, i.e., $b(t_1) \geqslant b(t_2)$ if $t_1 < t_2$,
- r is positive and $\int_0^\infty r(t)dt > b(0)$,
- and in any time interval (t_1, t_2) the price reduction is always smaller than the rental cost, i.e., $b(t_1) - b(t_2) < \int_{t_1}^{t_2} r(t)dt$.

We focus on the cost in the interval $[0, T]$.

a) [5] What is the optimal offline strategy, i.e., when T is known?

b) [6] Design a deterministic online algorithm with a competitive ratio of 2 for any $b(t)$ and $r(t)$.

c) [6] Assume b is a constant function, that is, the machine always costs the same. Now show a lower bound of the competitive ratio of deterministic algorithms.

d) [8] Suppose $b(t) = \max\{1000 - t, 200\}$ and $r(t) = 3$. Design a deterministic online algorithm with competitive ratio strictly smaller than 2.

Additional space for question 4

Additional space

1 Regular Languages (25 points)

1.1 NFA, DFA, REX [15 points]

Consider the language $L = \{w \mid w \in \{0,1\}^*$ such that w contains an even number of zeros or exactly two ones$\}$. For example, the strings 00011 and 111 belong to L, while the strings 100000 and 1110 do not. The empty string belongs to L.

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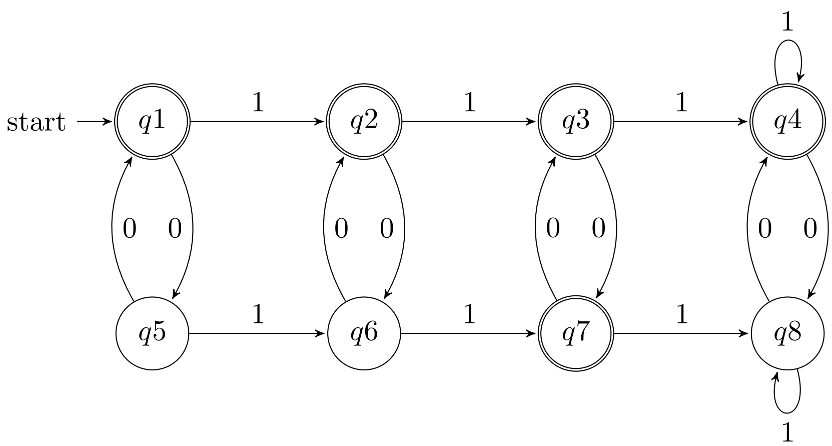
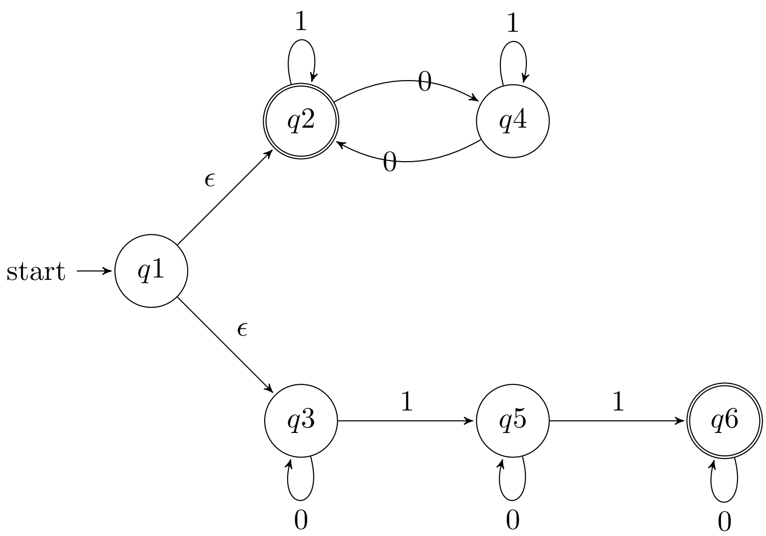
b) [6] Draw a DFA that recognizes L with 8 or less states.

c) [4] Give a Regular Expression for L.



a)



b)

c) $L = (1^*01^*0)^*1^* \cup 0^*\ 10^*10^*$

1.2 Regular or not? [10 points]

Are the following languages regular? If so, exhibit a finite automaton (deterministic or not) or a regular expression for it. If not, prove it formally using the pumping lemma or the closure properties of regular languages.

a) [5] $L = \{0^n \mid n \text{ is a power of } 2\}$.

b) [5] $L = \{w \mid w \in \{0,1\}^* \wedge \text{ there is no } x \text{ such that } w = xx\}$.

a) Let us assume that L is regular and show that this results in a contradiction.

We have seen that any regular language fulfills the pumping lemma. This means, there exists a number p, such that every word $w \in L$ with $|w| \geq p$ can be written as $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$, such that $xy^iz \in L$ for all $i \geq 0$.

In order to obtain the contradiction, we need to find at least one word $w \in L$ with $|w| \geq p$ that does not adhere to the above proposition. We choose $w = xyz = 1^{p^2}$ and consider the case $i = 2$ for which the Pumping Lemma claims $w' = xy^2z \in L$.

We can relate the lengths of $w = xyz$ and $w' = xy^2z$ as follows.

$$p^2 = |w| = |xyz| < |w'| = |xy^2z| \leq p^2 + p < p^2 + 2p + 1 = (p+1)^2$$

So we have $p^2 < |w'| < (p+1)^2$ which implies that $|w'|$ cannot be a square number since it lies between two consecutive square numbers. Therefore, $w' \notin L$ and hence, L cannot be regular.

b) Regular languages are closed under compliment. The compliment of L is $L' = \{w \mid w \in \{0,1\}^* \wedge$ there is a x such that $w = xx\}$ or $L = \{ww | w \in \{0,1\}^*\}$. The latter is not regular as proven in the lecture. Thus, also L is not regular.

1 Context-free languages (15 points)

1.1 Write me a grammar [5]

Give a context-free grammar (the production rules) for the following language:

$$L = \{a^n b^m \mid 0 \leq n \leq m \leq 3n\}$$

Solution

$$S \rightarrow aSb$$
$$S \rightarrow aSbb$$
$$S \rightarrow aSbbb$$
$$S \rightarrow \epsilon$$

1.2 Ambiguous or Not? [5]

Let G be the context-free grammar given by the following production rules:

$$S \to 0A \mid 1B$$
$$A \to 0AA \mid 1S \mid 1$$
$$B \to 1BB \mid 0S \mid 0$$

Is G ambiguous grammar? Explain your answer.

Solution
The grammar is *ambiguous*.
Example, deriving the string 001101:

$$\begin{array}{rrlr} S \to & 0A & S & \to 0A \\ \to & 00AA & & \to 00AA \\ \to & 001S1 & & \to 0011S \\ \to & 011B1 & & \to 0010A \\ \to & 001101 & & \to 001101 \end{array}$$

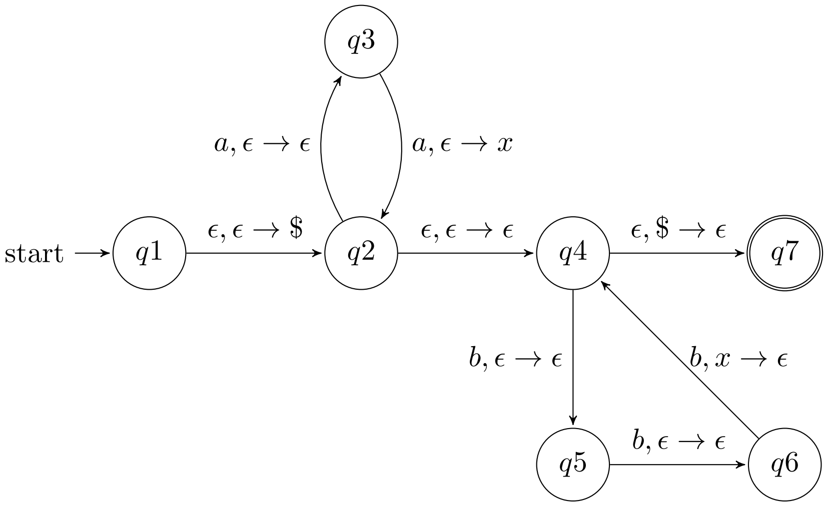
1.3 Draw me a PDA [5]

Consider the following Language:

$$L = \{a^{2n}b^{3n} \mid n \geq 0\}$$

Design a PDA that accepts L. The designed PDA must have 9 or *less* states.

Solution



1 King of the Hill (15 points)

King of the Hill is a game played between two players P1 and P2. The playground is a terrain with seven hills. There is a cage on top of some hill. After some random time (Poisson process with expected time 1 minute) the cage will disappear on the hill, and immediately appear again on another hill. The objective of the players is to be in the cage alone. Whenever a player is alone in the cage for time t, the player will earn t points. The time taken by P1 and P2 to get to the cage is a Poisson process with expected value 1/2 minute and 1 minute, respectively. When both players are inside the cage, no player will earn points, and there is a shootout between them. The shootout duration is a Poisson process and takes 1/2 minute on average. P1 wins the shootout with probability p and P2 with probability $1 - p$. The player who wins the shootout remains in the cage, whereas the other player again spawns somewhere in the playground, heading again for the cage.

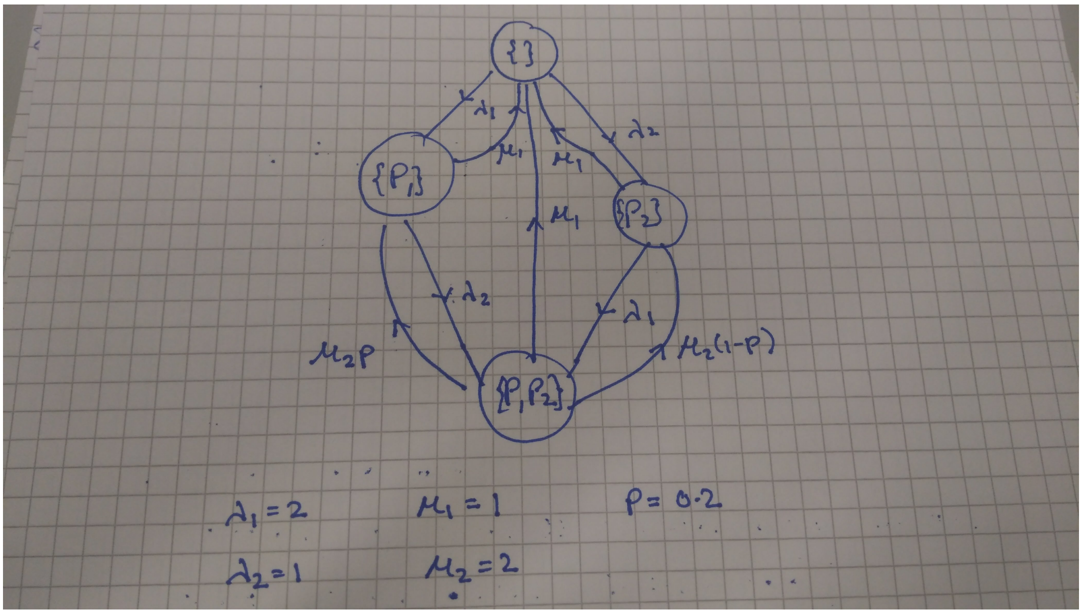
a) [5] Model the game as a Continuous Time Markov Chain.

b) [5] If the game is played long enough, what is the probability that the cage is empty?

c) [5] Assume P2 wins each and every shootout, i.e., $p = 0$. If the game is played long enough, which player gains more points in expectation?

Solution

a) See figure below.



b) Let π_0, π_1, π_2 and π_3 be the stationary state probabilities that the cage is empty, has P1, has P2, and both P1, P2. For the empty state, we have

$$\pi_0(\lambda_1 + \lambda_2) = (\pi_1 + \pi_2 + \pi_3)\mu_1. \tag{1}$$

As $\Sigma_{i=0}^{3}\pi_i = 1$, we can substitute $\pi_1 + \pi_2 + \pi_3$ for $1 - \pi_0$ in the above expression and get $\pi_0 = \frac{1}{4}$.

c) For the state $P1$, we have

$$2\pi_1 = 2\pi_0 + 0 \cdot 2\pi_3$$

Thus, $\pi_1 = \pi_0 = \frac{1}{4}$. For the state $\{P2\}$, we get

$$\begin{aligned} 3\pi_2 &= \pi_0 + 1 \cdot 2\pi_3 \\ 3\pi_2 &= \pi_1 + 2\pi_3. \end{aligned} \tag{2}$$

For the state $\{P1, P2\}$, we have

$$3\pi_3 = \pi_1 + 2\pi_2. \tag{3}$$

Using (2) and (3), we get $\pi_2 = \pi_3$. As $\Sigma_{i=0}^{3}\pi_i = 1$, we have $\pi_2 = \pi_3 = \frac{1}{4}$ as well. Thus, P1 and P2 have the same points in expectation.

1 GPU (25 points)

Alice needs a graphics processing unit (GPU) to do deep learning research. She can

- buy a GPU at time t for the price $b(t)$,
- or rent a GPU with rental cost rate $r(t)$: If Alice rents a GPU from time t_1 to t_2, then the total rental cost is $\int_{t_1}^{t_2} r(t)dt$.

Alice does not know the time (denoted by T) when she will not be interested in deep learning anymore. That is, after unknown time T, she does not need a GPU. Let's assume that

- $b : R_+ \mapsto R_+$ is a non-increasing function, i.e., $b(t_1) \geqslant b(t_2)$ if $t_1 < t_2$,
- r is positive and $\int_0^\infty r(t)dt > b(0)$,
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We focus on the cost in the interval $[0, T]$.

a) [5] What is the optimal offline strategy, i.e., when T is known?

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c) [6] Assume b is a constant function, that is, the machine always costs the same. Now show a lower bound of the competitive ratio of deterministic algorithms.

d) [8] Suppose $b(t) = \max\{1000 - t, 200\}$ and $r(t) = 3$. Design a deterministic online algorithm with competitive ratio strictly smaller than 2.

Solution

a) When $b(0) > \int_0^T r(t)dt$, always rent; or else, buy it. One may try to find the point to buy by optimizing the sum, but it is not necessary because of the third assumption.

b) buy it at time t_0 such that $b(t_0) = \int_0^{t_0} r(t)dt$.
There are two cases for the optimal offline strategy; either rent always or buy at $b(0)$. For both these cases we will show the competitive ratio is at most 2.

- Always rent: competitive ratio $r = \frac{\int_0^{t_0} r(t)dt + b(t_0)}{\int_0^T r(t)dt} \leqslant 2$
- Buy at $b(0)$: competitive ratio $r = \frac{\int_0^{t_0} r(t)dt + b(t_0)}{b(0)} = \frac{2b(t_0)}{b(0)} \leqslant \frac{2b(0)}{b(0)} \leqslant 2$, since $t_0 \geqslant 0$.

c) Whenever Alice buys a machine, she loses her interest in deep learning. Just need to analyse two cases, buy before t_0 or after t_0. Both give lower bounds at least 2.

- Alice buys at time $t_1 > t_0$: $r = \frac{\int_0^{t_1} r(t)dt + b}{b} > \frac{\int_0^{t_0} r(t)dt + b}{b} = \frac{2b}{b} = 2$
- Alice buys at time $t_2 < t_0$: $r = \frac{\int_0^{t_2} r(t)dt + b}{\int_0^{t_2} r(t)dt} = \frac{\int_0^{t_2} r(t)dt + \int_0^{t_0} r(t)dt}{\int_0^{t_2} r(t)dt} > \frac{2\int_0^{t_2} r(t)dt}{\int_0^{t_2} r(t)dt} = 2.$

d) Buy it at time point $500 > t > 250$. Any time point in between is ok. For example, if one buys at time 300, then the ratio is at most $(900 + 700)/1000 = 1.7$