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Distributed Computing

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Computational Thinking
Sample Solution to Exercise 14

1 Pattern Generation with Cellular Automata

The following 5 transistion rules form the desired pattern:

$$(white, white, white) \rightarrow white$$

$$(white, white, black) \rightarrow black$$
$$(black, white, white) \rightarrow black$$
$$(white, black, white) \rightarrow white$$
$$(black, white, black) \rightarrow white$$

2 PCP warm-up

a) Domino sequence 4-3-1-2 is a solution; both the upper and lower string is *bbabaaaabba*.

b) Any valid sequence can only begin with domino 1. Then the next character in the upper string has to be b, so the sequence can only continue with domino 3. At this point, we have the strings *abbaa* and *abbaaa*, so the next upper word must start with an a. However, neither of domino 1 and 2 are valid continuations: domino 1 gives the strings *abbaaab*/*abbaaaabb*, while domino 2 gives the strings *abbaaaaba*/*abbaaaabb*. Hence the problem has no solution.

c) Domino sequence 2-3-1-5-5 is a solution; both the upper and lower string is *bcacabbbbbbb*.

d) The only possible starting domino is 4, and this can only be followed by 2, giving us *dbc*/*dbca*. We can decide to continue this with either domino 1 or 5. If we select domino 5 (*dbcab*/*dbcabc*), this can only be followed by domino 3 (*dbcabc*/*dbcabca*); now the lower string again differs by a single letter a, so this essentially puts us back in the same situation. Hence after adding the subsequence $(5, 3)$ some number of times, we have to select domino 1 instead.

At this point we have the strings *...ad*/*...adda*. We can only extend this by domino 4, giving *...add*/*...addadb*. The only valid next step at this point is domino 1 again, giving *...addad*/*...addadbdda*. However, there is no way to continue this sequence, so there is no solution.

3 PCP variants

a) ab^* PCP is decidable. If we have a domino with $\alpha = \beta$, then the answers is clearly yes. Otherwise no two dominoes contains the same number of letters b. This means that if our sequence consists of at least two dominoes, then the second occurrence of the letter a cannot be at the same position in the upper and lower strings, so the two strings can never be identical, and hence the answer is no.

b) Limited-use PCP is decidable. Given n input dominoes, any valid sequence can consist of at most $n \cdot k$ dominoes, so there are at most $n^{n \cdot k}$ valid sequences. We can enumerate and check them all in finite time, so the problem is decidable.

c) Unique-triplet PCP is decidable. Since there are at most n^3 possible consecutive triplets we can form from n dominoes, and a sequence of length $n^3 + 2$ already contains n^3 consecutive triplets, any valid sequence can consist of at most $n^3 + 2$ dominoes. We can enumerate and check these in finite time.

d) Two-color PCP is undecidable. Consider a set of dominoes S for which PCP is undecidable; we know that such a set exists. Now let us take two copies of S, and color the dominoes in the first set red, and the dominoes in the second set blue. If the original problem with set S has a solution for PCP, then this set has a solution for two-color PCP: we can take any original PCP solution, and color the odd/even dominoes red/blue, respectively. On the other hand, if the original PCP problem has no solution, then neither does this new problem, since we are not even able to make the upper and lower strings identical. Hence solving this modified problem is equivalent to solving the original PCP on set S; this implies that our derived two-color problem is undecidable.

e) Half-used PCP is undecidable. Given a set of dominoes S for which PCP is undecidable, we can add another set S' of extra dominoes $|S'| = |S|$ such that the upper word α on these extra dominoes always contains a new symbol ξ_1, and the lower word β always contains a new symbol ξ_2. Since ξ_1 never occurs in a lower word (and similarly, ξ_2 never occurs in an upper word), no valid sequence can contain any of these extra dominoes; this implies that any valid sequence uses at most half of the available dominoes. Thus Half-used PCP with this new set is solvable if and only if PCP is solvable on set S, making Half-used PCP undecidable.

f) Silly PCP is decidable. If we have a domino with $\alpha = \beta$, then the answers is clearly yes. Otherwise, the first domino of the sequence has distinct words (α, β) of the same length k. This means that one of the first k letters of the upper and lower strings will be different in any case, so no solution is possible.

g) Almost-silly PCP is undecidable. Given a set of dominoes S for which PCP is undecidable, we can take every letter on every domino in S, and replace it by c consecutive copies of the letter, essentially only switching to a "longer representation" of each symbol. This new problem satisfies the divisibility condition and it is solvable exactly if the original PCP problem was solvable; this implies undecidability.

h) Binary PCP is undecidable. Given a set of dominoes S for which PCP is undecidable, over an alphabet of $|\Sigma|$ characters, we can select a binary representation for each of these characters on $\lceil \log |\Sigma| \rceil$ bits, and replace the character on each domino by this binary representation. Once again this new problem has a valid domino sequence if and only if the original problem has a valid sequence, so it is undecidable.