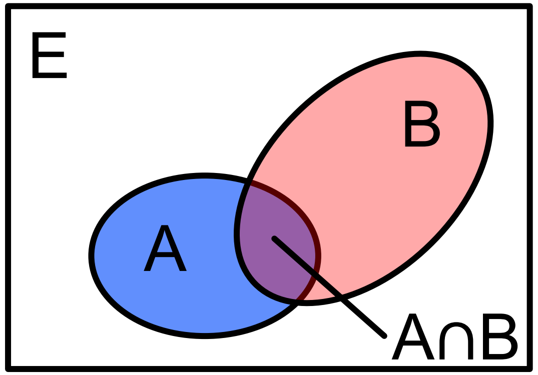
Crash course – Verification of Finite Automata
Binary Decision Diagrams

Xiaoxi He

Equivalence of representations



Sets

- Set algebra
- $\cup$, $\cap$, $\neg$

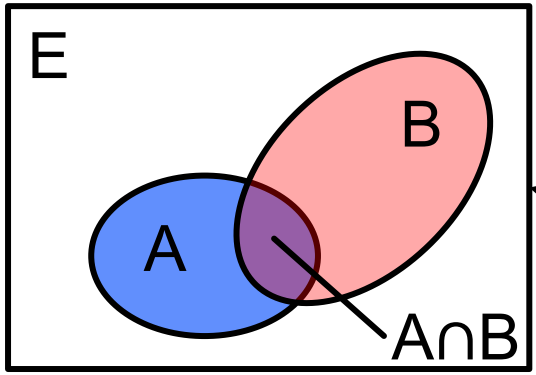
$\Leftrightarrow$

$$\psi_E = 1$$
$$\psi_A = f$$
$$\psi_B = g$$
$$\psi_{A \cap B} = f \cdot g$$

Boolean functions/
Characteristic functions

- Boole algebra
- $+$, $\cdot$, $\neg$

Equivalence of representations



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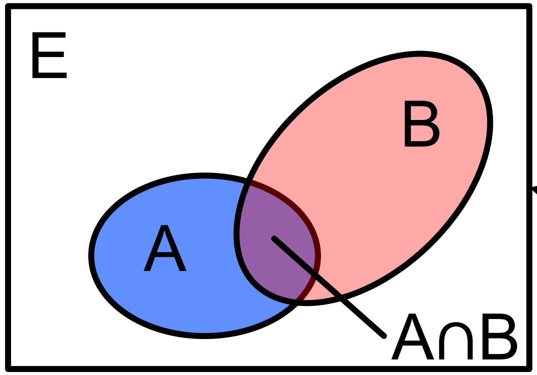
Binary encoding:

$$\sigma: X \rightarrow \Psi$$

→ Map a state to a binary expression

→ Bijective mapping!

Equivalence of representations



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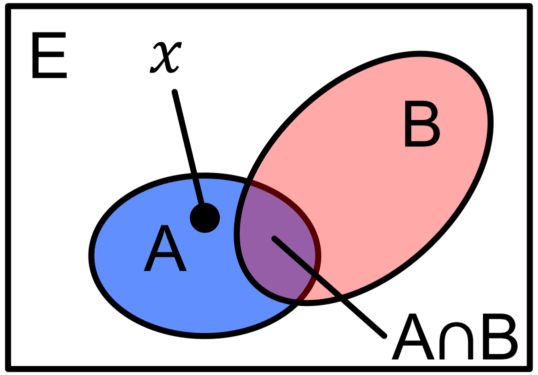
States

Boolean functions

→ Map a state to a binary expression

→ Bijective mapping!

Equivalence of representations



Sets

$\Leftrightarrow$

$$\psi_E = 1$$
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Boolean functions/
Characteristic functions

- A
- $s \in A$ (proposition)

$\longleftrightarrow$ $\sigma(\cdot)$ $\longleftrightarrow$

- ψ_A
- $\psi_A(\sigma(s)) = 1$
  $\sigma(s) \vDash \psi_A$
  or just $s \vDash \psi_A$

Reads "s satisfies ψ_A"

Example:

$\sigma(s) = x_1\overline{x_0} = (1,0)$ and $\psi_A = x_1 + x_0$

$\rightarrow s \vDash \psi_A$?

Binary Decision Diagrams

Based on the Boole-Shannon decomposition:

$$\underline{f} = \overline{x} \cdot \underline{f\,|_{x=0}} + x \cdot \underline{f\,|_{x=1}}$$

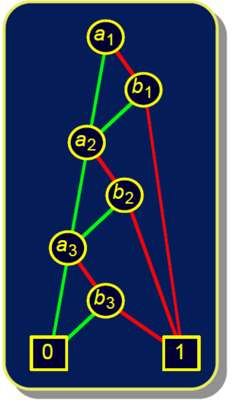
Boolean function of n and $(n-1)$ variables

→ For a given order of variable, **the decomposition is unique!**

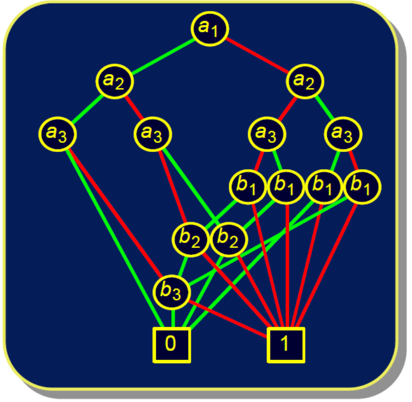
→ Hence the uniqueness of R(reduced)O(rdered)BDD.

Reminder:

In practice, simplicity of BDD depends strongly on the order.



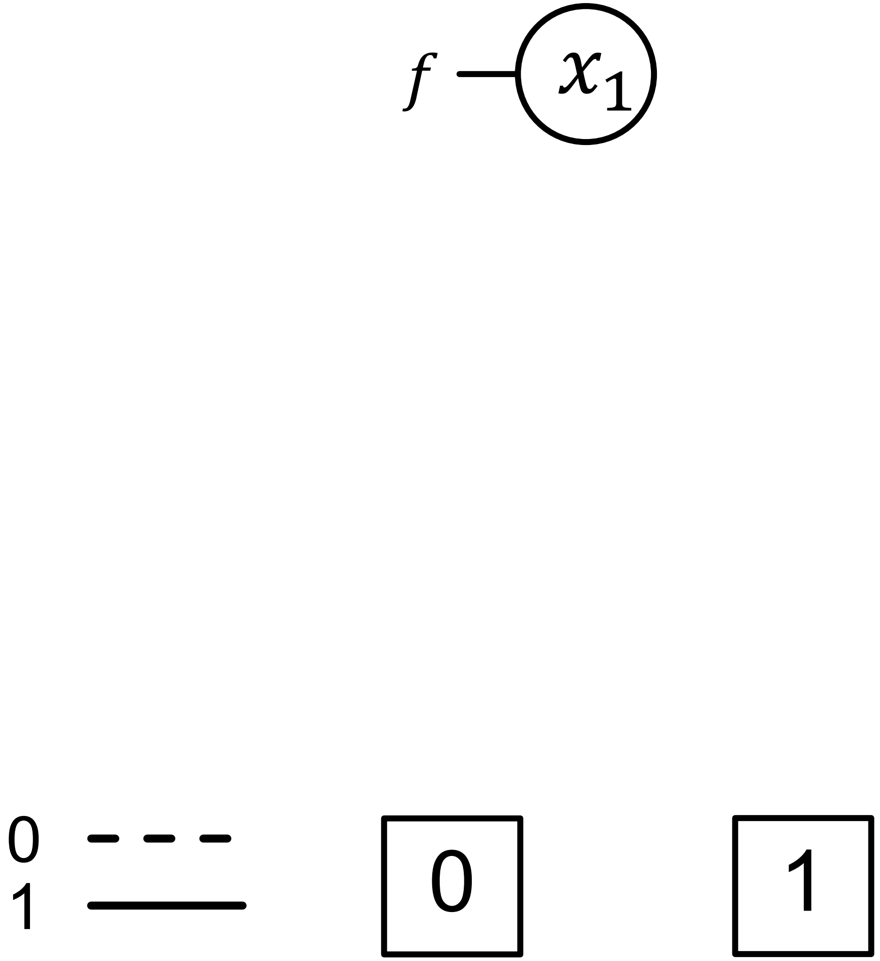
$(a_1 \wedge b_1) \vee$
$(a_2 \wedge b_2) \vee$
$(a_3 \wedge b_3)$



Good vs. Bad ordering

Binary Decision Diagrams: an example

$f: x_1 + \overline{x_1}\, x_2 + \overline{x_2}\, \overline{x_3}$

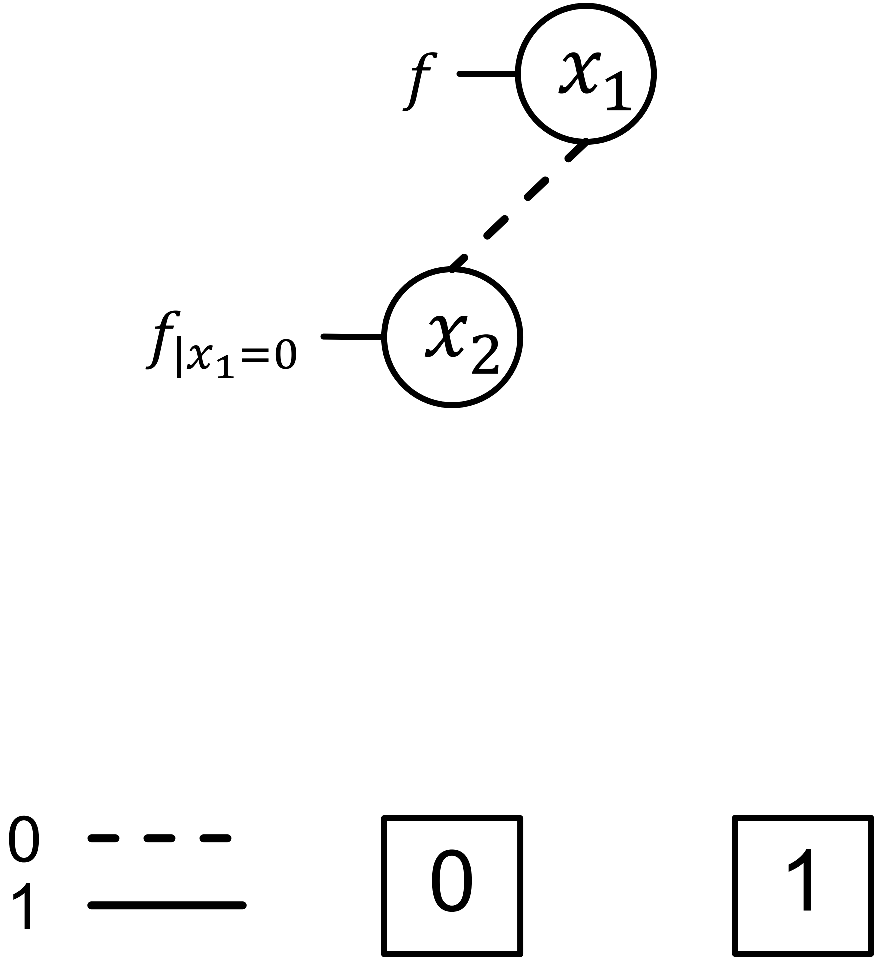


Binary Decision Diagrams: an example

$f: x_1 + \overline{x_1}\, x_2 + \overline{x_2}\, \overline{x_3}$

Fall $x_1 = 0$

$f_{|x_1=0}: x_2 + \overline{x_2}\, \overline{x_3}$



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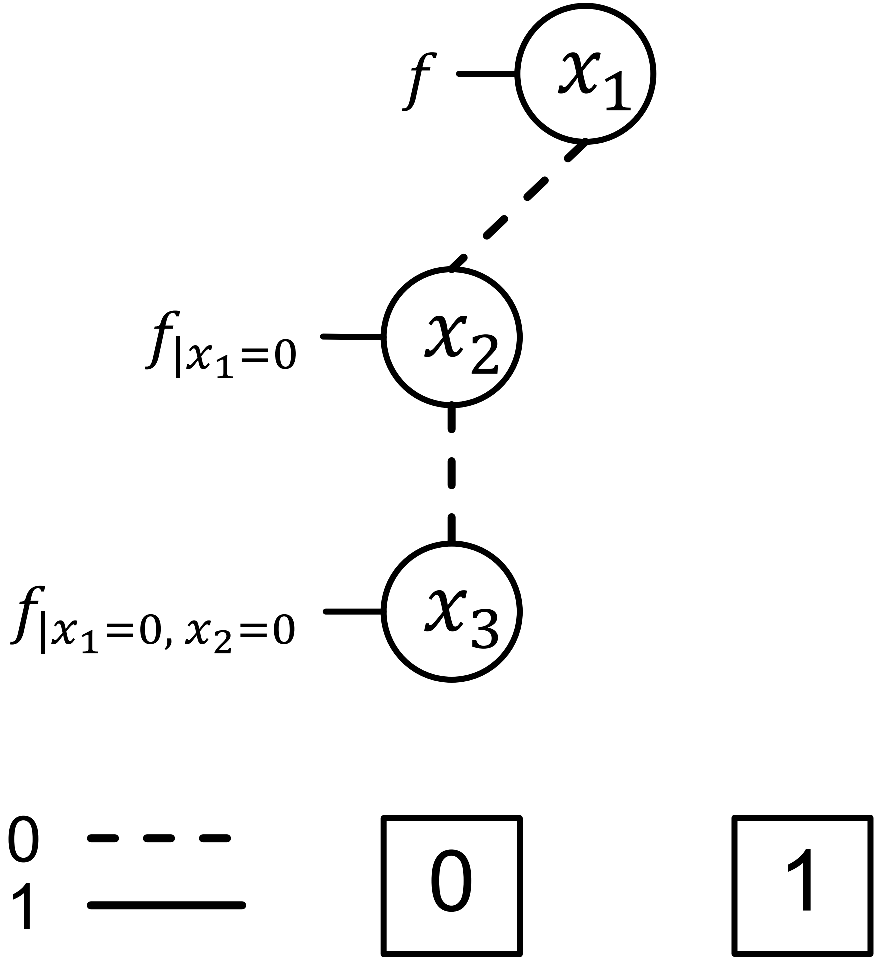
$f_{|x_1=0,\, x_2=0}: \overline{x_3}$



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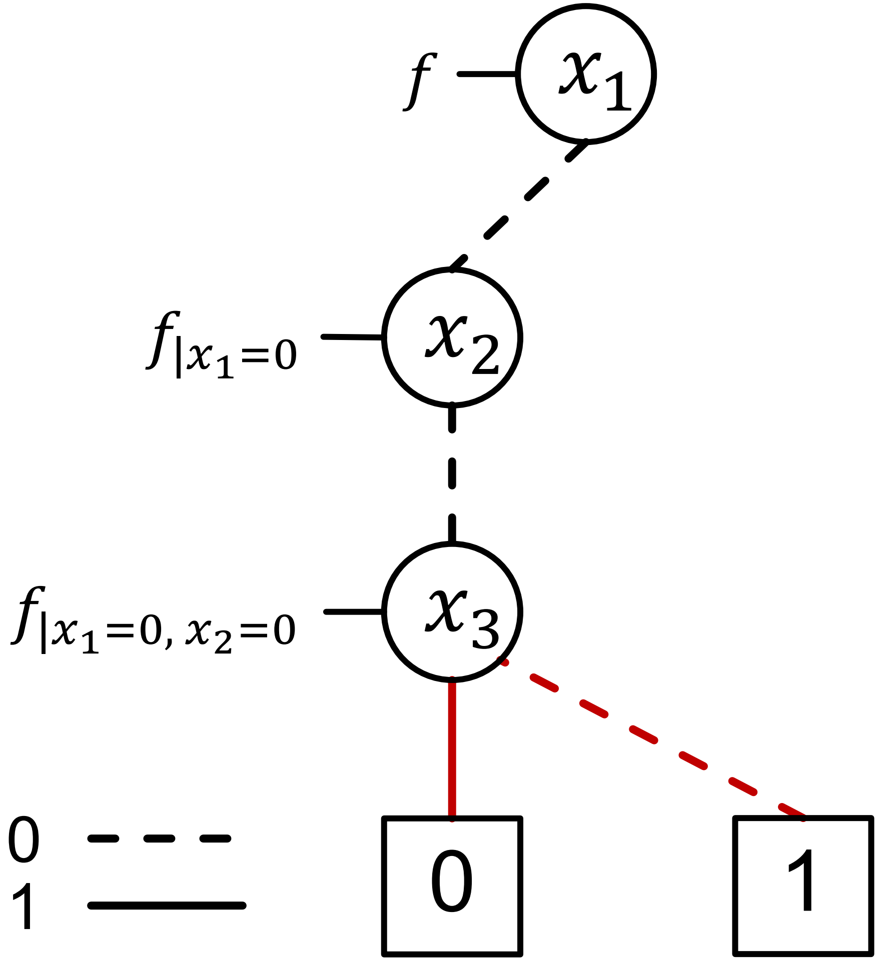

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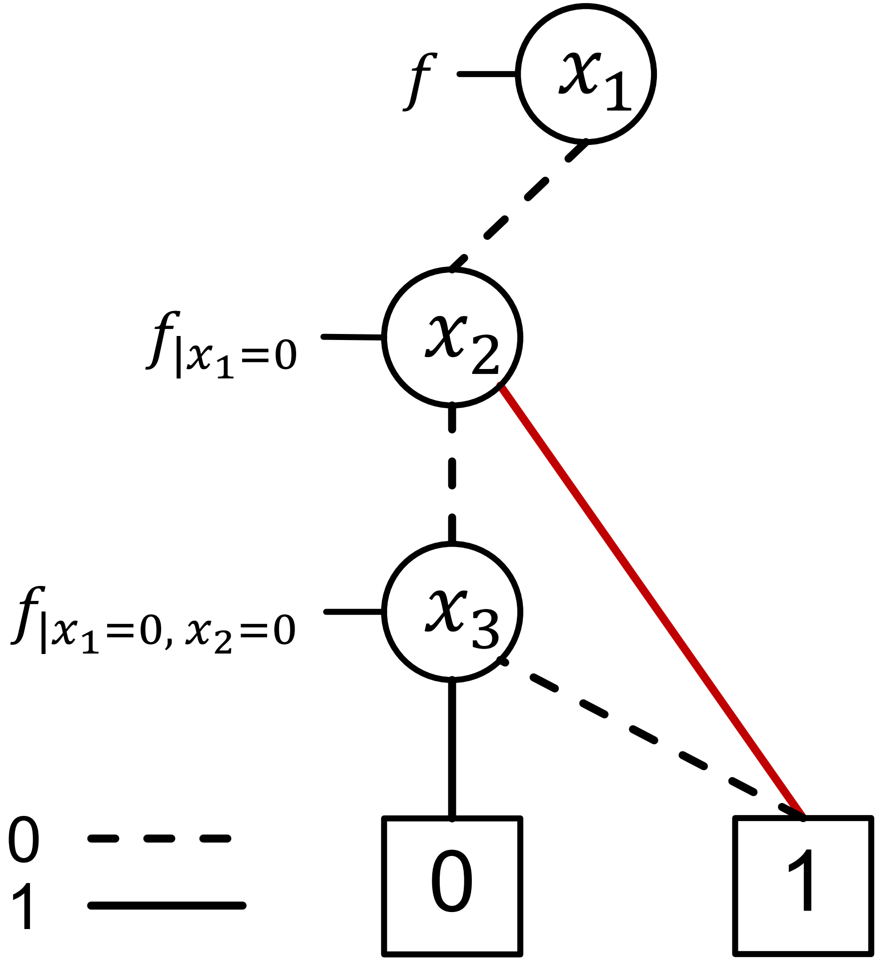
Fall $x_2 = 1$

$f_{|x_1=0,\, x_2=1}: 1$



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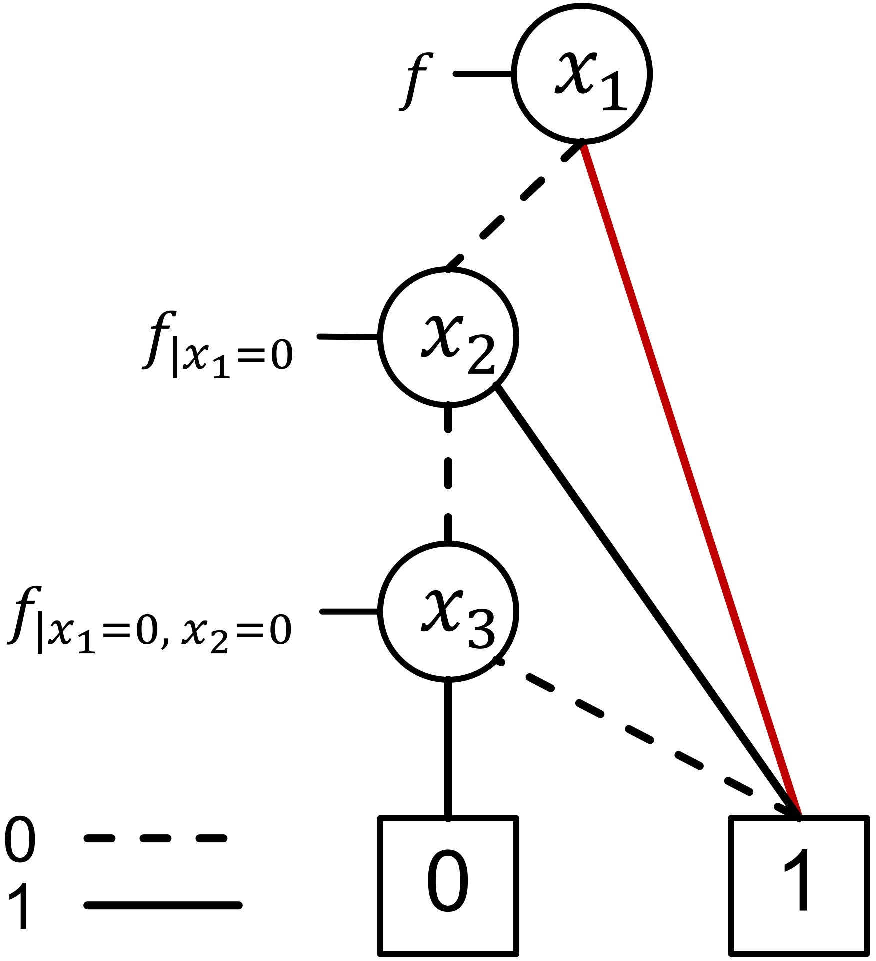

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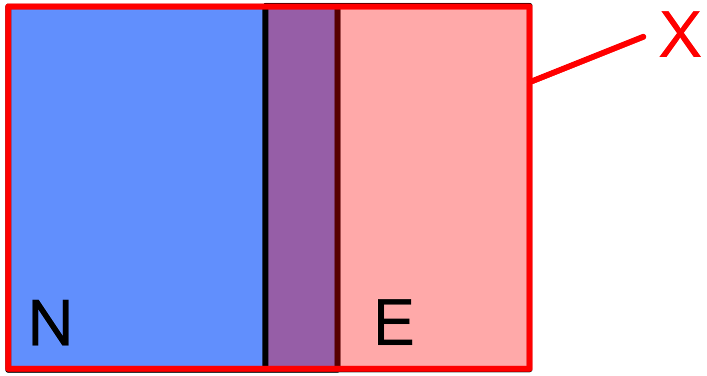
Binary Decision Diagrams

Your turn !

Ex1: Sets Representation

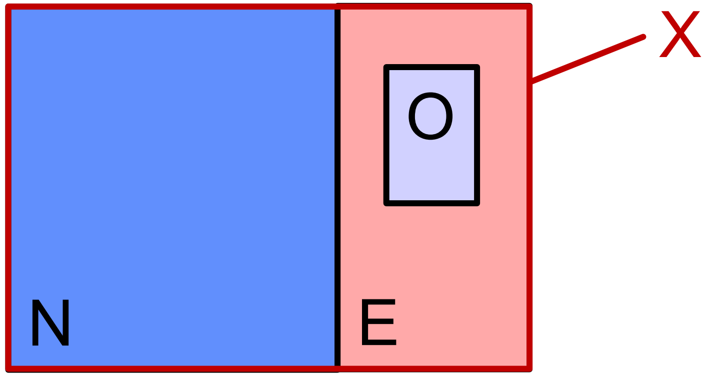
“Each state is either a nominal or an error state or both".



$\Rightarrow$ $$N \cup E = X \quad \Leftrightarrow \quad \psi_N + \psi_E = 1$$

Ex1: Sets Representation

"If a state is in the overflow set, it is not a nominal state".

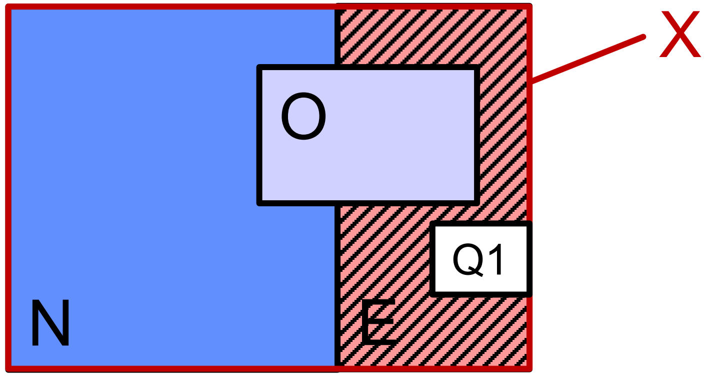


$\Rightarrow \qquad N \cap O = \emptyset \quad \Leftrightarrow \quad \psi_N \cdot \psi_O = 0$

But note it is not necessarily true !!
Although you would like it to be…

Ex1: Sets Representation

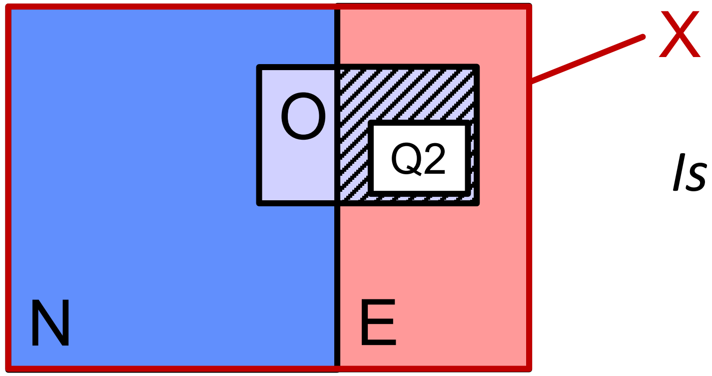
Describe Q1, the set of error states which are not an overflow, in term of sets and characteristic functions.



$\Rightarrow$ $$Q_1 = E \setminus O \quad \Leftrightarrow \quad \psi_{Q_1} = \psi_E \cdot \overline{\psi_O}$$

Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.



Is that correct ?

Ex1: Sets Representation

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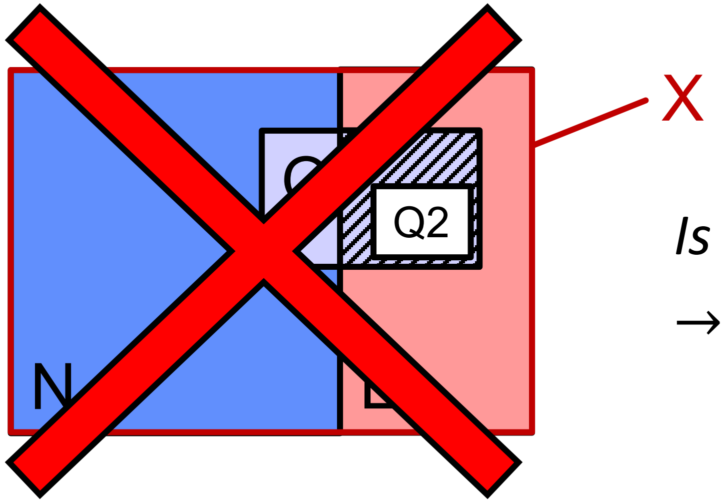

Is that correct? No!

→ *What if a state is not in O?*

Property is always true!

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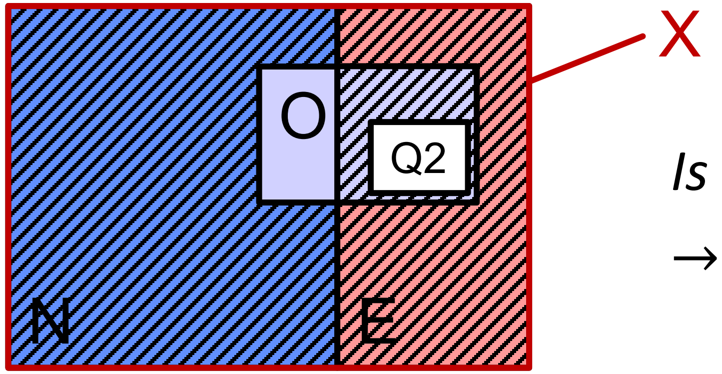

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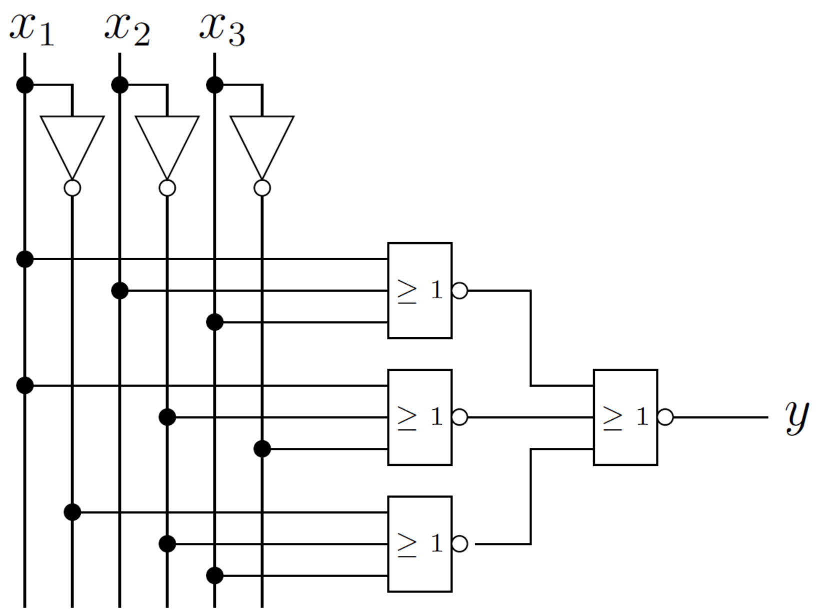
→ *What if a state is not in O?*

Property is always true!

⇒

$$Q_2 = (O \cap E) \cup \overline{O} = (O \cup \overline{O}) \cap (E \cup \overline{O})$$
$$= X \cap (E \cup \overline{O})$$
$$= E \cup \overline{O} \quad \Leftrightarrow \quad \psi_{Q_2} = \psi_E + \overline{\psi_O}$$

Ex2.1 Verication using BDDs



a) $f_2 : y = \overline{\overline{x_1 + x_2 + x_3} + \overline{x_1 + \overline{x_2} + \overline{x_3}} + \overline{\overline{x_1} + \overline{x_2} + x_3}}$

Ex2.1 Verication using BDDs

$$f_1 : (x_1\overline{x_2} + x_1x_3 + \overline{x_2}x_3 + \overline{x_1}x_2\overline{x_3})$$

$Fall\ x_1 = 0$

$y_{|x_1=0} = \overline{x_2}x_3 + x_2\overline{x_3}$

$Fall\ x_2 = 0$

$y_{|x_1=0,x_2=0} = x_3$

$Fall\ x_2 = 1$

$y_{|x_1=0,x_2=1} = \overline{x_3}$

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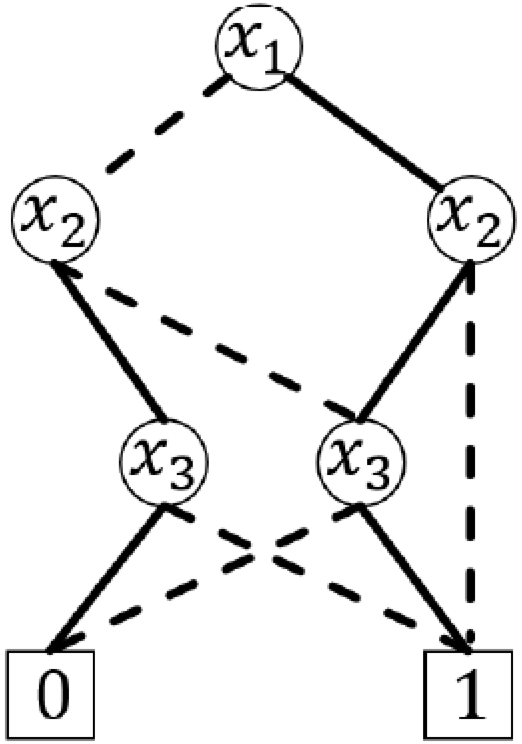

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$Fall\ x_1 = 0$

$$y_{|x_1=0} = \overline{\overline{x_2 + x_3} + \overline{\overline{x_2} + \overline{x_3}}}$$

$Fall\ x_2 = 0$

$$y_{|x_1=0,x_2=0} = \overline{\overline{x_3} + \overline{1 + \overline{x_3}}} = x_3$$

$Fall\ x_2 = 1$

$$y_{|x_1=0,x_2=1} = \overline{\overline{1} + \overline{\overline{x_3}}} = \overline{x_3}$$

$Fall\ x_1 = 1$

$$y_{|x_1=1} = \overline{\overline{1} + \overline{1} + \overline{\overline{x_2} + x_3}} = \overline{x_2} + x_3$$

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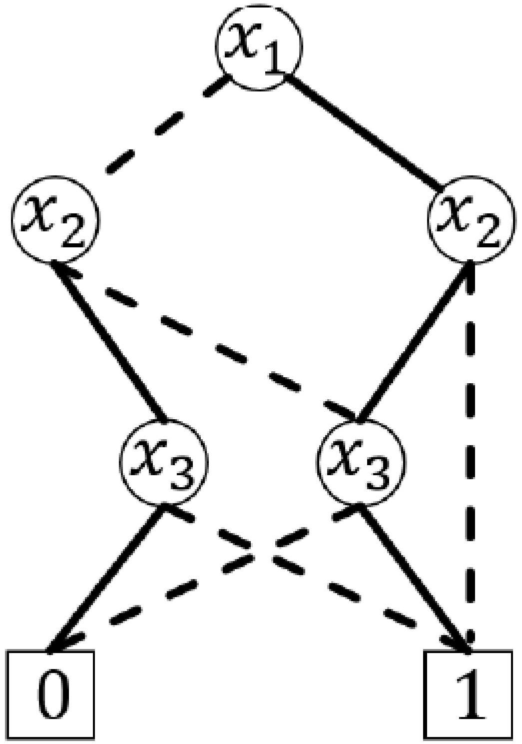
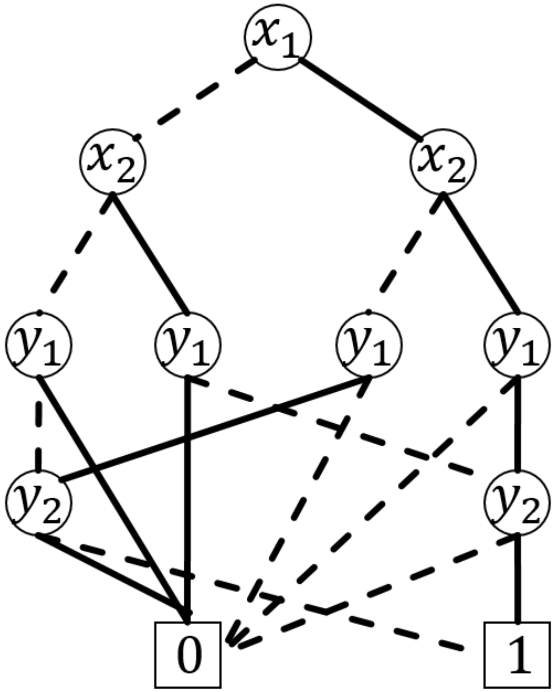


Ex2.2 BDDs with respect to different orderings

$$g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2) \;,\quad \Pi : x_1 < x_2 < y_1 < y_2$$

a) $g = x_1\{x_2[y_1(y_2) + \overline{y_1}(0)] + \overline{x_2}[y_1(\overline{y_2}) + \overline{y_1}(0)]\}$
$\quad + \overline{x_1}\{x_2[y_1(0) + \overline{y_1}(y_2)] + \overline{x_2}[y_1(0) + \overline{y_1}(\overline{y_2})\}$

b)



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$$\textbf{c)} \quad g = x_1\{y_1[x_2(y_2) + \overline{x_2}(\overline{y_2})] + \overline{y_1}[0]\} + \overline{x_1}\{y_1[0] + \overline{y_1}[x_2(y_2) + \overline{x_2}(\overline{y_2})]\}$$

Better ordering:

6 vs. 9 nodes