Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Distributed Computing

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Distributed Systems Part II

Solution to Exercise Sheet 5

1 The Resilience of a Quorum System

a) No such quorum system exists. According to the definition of a quorum system, every two quorum of a quorum system intersect. So at least one server is part of both quorums. The fact that all servers of a particular quorum fail, implies that in each other quorum at least one server fails, namely the one which lies in the intersection. Therefore it is not possible to achieve a quorum anymore and the quorum system does not work anymore.

b) Just 1 - as soon as 2 servers fail, no quorum survives.

c) Imagine a quorum system in which all quorums overlap exactly in one single node. Each element of the powerset of the remaining $n-1$ nodes joined with this special node is a quorum. This gives 2^{n-1} quorums.
Can there be more? No! Consider a set from the powerset of n servers. Its complement cannot be a quorum as well, as they do not overlap. So, from each such couple, at most one set can be part of the quorum system. This gives an upper bound of $2^n/2 = 2^{n-1}$.

2 A Quorum System

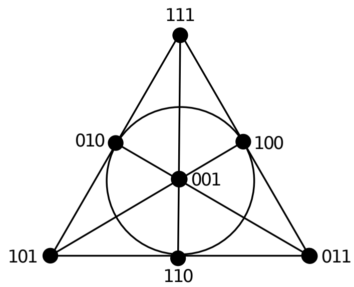

Figure 1: Quorum System

a) This quorum system consists of 7 quorums. As work is defined as the minimum (over all access strategies) expected number of servers in an accessed quorum, this system's work is 3 (all strategies induce the same work on a system where all quorums are the same size). The best access strategy consists of uniformly accessing each quorum (you will prove this for a more general case in exercise 3), so its load is 3/7.

b) Its resilience $R(\mathcal{S}) = 2$. Proof: every node is in exactly 3 quorums, so 2 nodes can be contained in at most $2 \cdot 3 = 6 < 7 = |\mathcal{S}|$ quorums, thus if no more than 2 nodes fail, there will be at least 1 quorum without a faulty node. If on the other hand for example the nodes 101, 010 and 111 fail, no other quorum can be achieved; see also exercise 1a).

3 S-Uniform Quorum Systems

Definitions:

s-uniform: A quorum system $\mathcal{S}$ is *s-uniform* if every quorum in $\mathcal{S}$ has exactly s elements.
Balanced access strategy: An access strategy Z for a quorum system $\mathcal{S}$ is *balanced* if it satisfies $L_Z(v_i) = L$ for all $v_i \in V$ for some value L.

Claim: An s-uniform quorum system $\mathcal{S}$ reaches an optimal load with a balanced access strategy, if such a strategy exists.

a) In an s-uniform quorum system each quorum has exactly s elements, so independently of which quorum is accessed, s servers have to work. Summed up over all servers we reach a total load of s, which is the work of the quorum system. As the load induced by an access strategy is defined as the maximum load on any server, the best strategy is to evenly distribute this work on all servers.

b) Let $V = \{v_1, v_2, ..., v_n\}$ be the set of servers and $\mathcal{S} = \{Q_1, Q_2, ..., Q_m\}$ an s-uniform quorum system on V. Let Z be an access strategy, thus it holds that: $\sum_{Q \in \mathcal{S}} P_Z(Q) = 1$. Furthermore let $L_Z(v_i) = \sum_{Q \in \mathcal{S}; v_i \in Q} P_Z(Q)$ be the load of server v_i induced by Z.

Then it holds that:

$$\sum_{v_i \in V} L_Z(v_i) = \sum_{v_i \in V} \sum_{Q \in \mathcal{S}; v_i \in Q} P_Z(Q) = \sum_{Q \in \mathcal{S}} \sum_{v_i \in Q} P_Z(Q)$$
$$= \sum_{Q \in \mathcal{S}} P_Z(Q) \sum_{v_i \in Q} 1 \stackrel{*}{=} \sum_{Q \in \mathcal{S}} P_Z(Q) \cdot s = s \cdot \sum_{Q \in \mathcal{S}} P_Z(Q) = s$$

The transformation marked with an asterisk uses the uniformity of the quorum system.

To minimize the maximal load on any server, the optimal strategy is to evenly distribute this load on all servers. Thus if a balanced access strategy exists, this leads to a system load of s/n.

Note: A balanced access strategy does not exist for example for the following 2-uniform quorum system: $V = \{1, 2, 3\}$, $\mathcal{S} = \{\{1, 2\}, \{1, 3\}\}$. We have $\min\{L_Z(2), L_Z(3)\} < L_Z(1) = 1$ for any access strategy on this system.