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based on Prof. R. Wattenhoffer's material

Discrete Event Systems

Solution to Exercise Sheet 2

1 Pumping Lemma [Exam]

The Pumping Lemma in a Nutshell

Given a language L, assume for contradiction that L is regular and has the pumping length p. Construct a suitable word $w \in L$ with $|w| \geq p$ ("there *exists* $w \in L$") and show that for *all* divisions of w into three parts, $w = xyz$, with $|x| \geq 0$, $|y| \geq 1$, and $|xy| \leq p$, there *exists* a pumping exponent $i \geq 0$ such that $w' = xy^iz \notin L$. If this is the case, L is not regular.

Language L_1 can be shown to be non-regular using the pumping lemma. Assume for contradiction that L_1 is regular and let p be the corresponding pumping length. Choose w to be the word 0110^p1^p. Because w is an element of L_1 and has length more than p, the pumping lemma guarantees that w can be split into three parts, $w = xyz$, where $|xy| \leq p$ and for any $i \geq 0$, we have $xy^iz \in L_1$. In order to obtain the contradiction, we must prove that for every possible partition into three parts $w = xyz$ where $|xy| \leq p$, the word w cannot be pumped. We therefore consider the various cases.

a) If y starts anywhere within the first three symbols (i.e. 011) of w, deleting y (pumping with $i = 0$) creates a word with an illegal prefix (e.g. $1\,0^p\,1^p$ for $y = 01$).

b) If y consists of only 0s from the second block, the word $w' = xy^2z$ has more 0s than 1s in the last $|w'| - 3$ symbols and hence $c \neq d$.

Note that y cannot contain 1s from the second block because of the requirement $|xy| \leq p$.

We have shown that for all possible divisions of w into three parts, the pumped word is not in L_1. Therefore, L_1 cannot be regular and we have a contradiction.

Be Careful!

The argumentation above is based on the closure properties of regular languages and only works in the direction presented. That is, for an operator $\diamond \in \{\cup, \cap, \bullet\}$, we have:

If L_1 and L_2 are regular, then $L = L_1 \diamond L_2$ is also regular.

If either L_1 or L_2 or both are non-regular, we cannot deduce the non-regularity of L or vice-versa. Moreover, L being regular does not imply that L_1 and L_2 are regular as well. This may sound counter-intuitive which is why we give examples for the three operators.

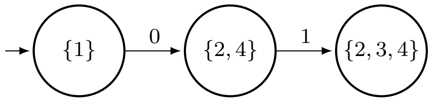
- $L = L_1 \cup L_2$: Let L_1 be any non-regular language and L_2 its complement. Then $L = \Sigma^*$ is regular.

- $L = L_1 \cap L_2$: Let L_1 be any non-regular language and L_2 its complement. Then $L = \emptyset$ is regular.
- $L = L_1 \bullet L_2$: Let $L_1 = \{a^*\}$ (a regular language) and $L_2 = \{a^p \mid p \text{ is prime}\}$ (a non-regular language) then $L = \{aaa^*\}$ is regular.

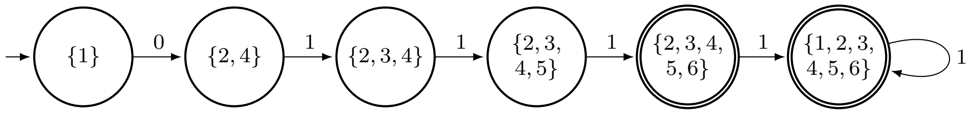
Hence, to prove that a language L_x is non-regular, you assume it to be regular for contradiction. Then you combine it with a *regular* language L_r to obtain a language $L = L_x \diamond L_r$. If L is non-regular, L_x could not have been regular either.

2 Deterministic Finite Automata [Exam]

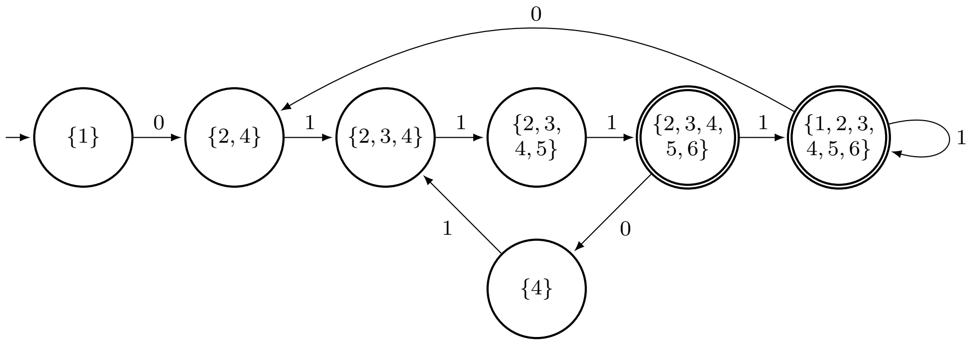
We could use the systematic transformation scheme presented in the lecture (slide 1/75). Considering the large number of states, however, this will easily lead to an explosion of states in the derandomized automaton. Hence, we build the deterministic finite automaton in a step-wise manner, only creating those states that are actually required: Initially, the automaton requires a 0. Subsequently, only a 1 is accepted. Including the various transitions, this 1 can lead to three different states, namely states 2, 3, and 4.



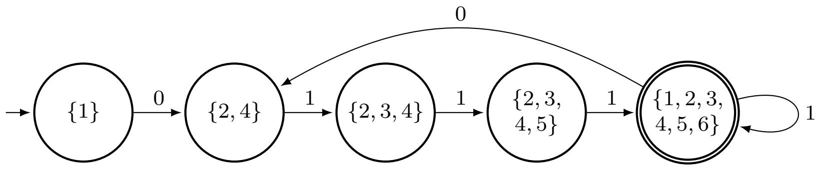
In any of the states 2, 3, and 4, only a 1 is accepted. Assume that the automaton is currently in state 2, this 1 can lead to states $\{2, 3, 4\}$ when including all ε-transitions. When in state 3, the 1 leads to states $\{2, 3, 4, 5\}$ and finally, when being in state 4, the reachable states given a 1 are $\{2, 3, 4\}$. Hence, a 1 leads from state $\{2, 3, 4\}$ to state $\{2, 3, 4, 5\}$. Repeating the same process for state $\{2, 3, 4, 5\}$, we can see that, again, only a 1 is accepted, which leads to state $\{2, 3, 4, 5, 6\}$. Because the state 6 in the original NFA was an accepting state, $\{2, 3, 4, 5, 6\}$ is also accepting in the DFA. From state $\{2, 3, 4, 5, 6\}$, an additional 1 will lead to another accepting state $\{1, 2, 3, 4, 5, 6\}$. And from this state, any subsequent 1 returns to state $\{1, 2, 3, 4, 5, 6\}$ as well.



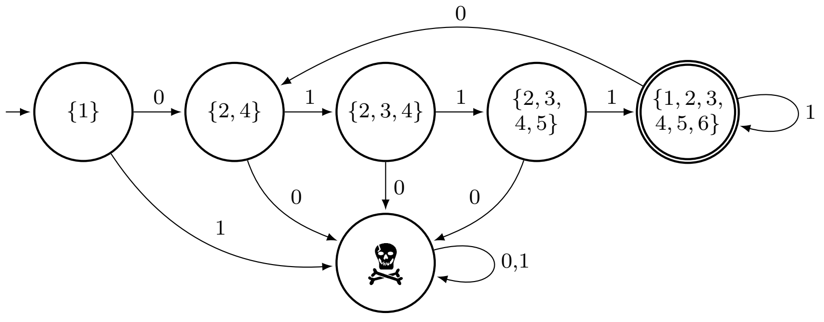
What happens if a 0 occurs in the input? This is feasible only when the deterministic state includes either state 1 or state 6. In state $\{2, 3, 4, 5, 6\}$, a 0 necessarily leads to state $\{4\}$, whereas in state $\{1, 2, 3, 4, 5, 6\}$ a 0 leads to state $\{2, 4\}$. In both of these states, the only acceptable input symbol is a 1 and leads to the state $\{2, 3, 4\}$. Hence, the deterministic finite automaton looks like this:



It can easily be seen, that first the states $\{4\}$, $\{2,4\}$ and then the states $\{2,3,4,5,6\}$, $\{1,2,3,4,5,6\}$ can be merged and hence, the automaton can be reduced to the one shown in the next figure.

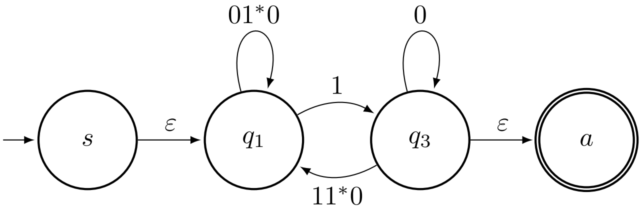


This is not a DFA yet, because the crash state is still missing. The final deterministic automaton looks like this:

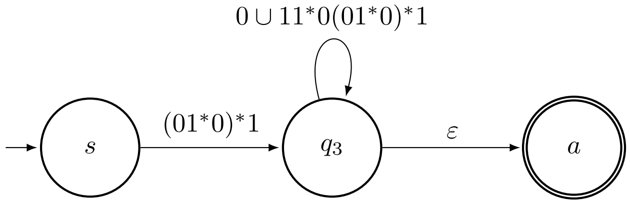


3 Transforming Automata [Exam]

The regular expression can be obtained from the finite automaton using the transformation presented in the script on slide 1/85. After ripping out state q_2, the corresponding GNFA looks like this:



After also removing state q_1, the GNFA looks as follows.



Eliminating the last state q_3 yields the final solution, which is $(01^*0)^*1(0 \cup 11^*0(01^*0)^*1)^*$.

Note: Ripping out the interior states in a different order yields a distinct yet equivalent regular expression. The order q_3, q_2, q_1, for example, results in $((0 \cup 10^*1)1^*0)^*10^*$.

4 Regular and Context-Free Languages

a) Sometimes, even simple grammars can produce tricky languages. We can interpret the 1s and 2s of the second production rule as opening and closing brackets. Hence, $L(G)$ consists of all correct bracket terms where at least one 0 must be in each bracket.

Choose $w = 1^p 0 2^p \in L(G)$. Let $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$ (pumping lemma). Because of $|xy| \leq p$, xy can only consist of 1s. According to the pumping lemma, we should have $xy^i z \in L$ for all $i \geq 0$. However, by choosing $i = 0$ we delete at least one 1 and get a word $w' = 1^{p-|y|} 0 2^p$ with $|y| \geq 1$. w' is not in L since it has fewer 1s than 2s. This means that w is not pumpable and hence, $L(G)$ is not regular.

b) Since *every* regular language is also context-free, we can choose an arbitrary regular language. For example, we can choose the language $L = \{0^n 1, n \geq 1\}$ which is clearly regular. A context-free grammar for this language uses only the production $S \to 0S \mid 1$.