Discrete Event Systems
Exercise Sheet 7

1   Bin Packing

In order to finance their living, Darya and Sebastian work at the assembly line of a production firm. Since they usually are physically stressed from trying to obtain a PhD, their task is very simple: They have to pick the items delivered on the assembly line, put them into a bin and close the bin.

Assume that there are n items of size $s_i \leq 1$ while the bins have size 1. Moreover, assume that the algorithm used by Darya and Sebastian is a very simple one: The items are handled in order of their arrival and put into a bin as long as there is enough space left. If an item arrives that does not fit into the bin anymore, they close the bin and start with a new empty bin. Calculate the competitive ratio with respect to the total number of bins Darya and Sebastian need compared to an offline algorithm which distributes the items optimally among the bins.

2   Paging

Paging plays an important role in almost every computer system. Typically, there is a fast cache which allows fast access, but has limited space. On the other hand, access to the disk is slow, but space is plentiful.

We consider a simple system in which the cache has enough space to store three pages. Given a request for a page p_i, the system must make p_i available in the cache. If p_i is already in the cache (called a *hit*), the system does nothing. Otherwise (a *miss*), the system incurs a *page fault* and must copy the page p_i from the disk to one of the three locations in the cache. In case all three slots in the cache are already occupied with other pages, the system is faced with the problem of which page to evict from the cache in order to make space for p_i.

In our model, we have to pay a price of 1 for each page fault, while accessing a page that is already in the cache is for free. In this exercise, we analyze the competitiveness of several well known paging strategies.

a) Which of the following paging strategies are constant-competitive? What is the corresponding competitive ratio?
   Remark: An online strategy is called *constant-competitive* if its competitive ratio is bounded by a constant.

   - LFU (Least Frequently Used): Replace the page that has been requested the smallest number of times since entering the fast memory.
   - LIFO (Last-in/First-out): Replace the page most recently moved to the cache.
   - FIFO (First-in/First-out): Replace the page that has been in the cache longest.
   - LRU (Least Recently Used): When eviction is necessary, replace the page whose most recent request was the earliest.

- Fwf (Flush When Full): Whenever there is a page fault and there is no space left in the cache, evict *all* pages currently in the cache.

b) All the above strategies are deterministic. Prove a lower bound on the competitive ratio of any deterministic paging strategy.

3 Memory

Memory (or Concentration, Pairs, Pelmanism, Pexeso, Shinkei-suijaku) is a popular card game that requires good memorization skills. A deck of pairs of cards is shuffled and laid out face down on a table. The goal of the game is to collect all pairs of cards: In the solitaire version, in as few moves as possible, and with multiple players, to collect more pairs than each opponent. In a single move, a player may first turn over a single card and then another card – if both cards form a pair, they are collected, else they are turned over again. While children often have an advantage in the game due to their innate memory skills, professional tournament games with 32 or 31 pairs are often played by using strategies with a quite sophisticated mathematical background.

The game starts with $2n$ cards laid out face down on a table, so that a player can only see the identical back sides. W.l.o.g. each card is labeled with a natural number from 1 to n on its front side, with each label appearing exactly twice. Two cards with the same label form a pair. The $2n$ cards are shuffled uniformly at random beforehand, but the position of each uncollected card stays fixed during the game once it is laid out.

Again, the sole player makes a move as follows: first she turns over one card, then another card. If both cards have the same label, then the pair is collected, else both cards are put back into the same position face down. The goal of the game is to collect all pairs in as few moves as possible. We assume that the player has perfect memorization capabilities, meaning that she can remember all moves and the corresponding cards and their positions. We assume that if the player has collected a pair in a move, then her next move is not free.

If one considers the solitaire Memory game as an online problem, then a strategy competes against a fictional optimal offline player that already knows the labels of the cards beforehand. With all information available, a pair can be collected in each move by the offline player, requiring n moves in total.

a) Describe a deterministic strategy that is 2–competitive for the solitaire version of memory.

b) Can a deterministic strategy be better than 2–competitive? Also show a good lower bound for the competitive ratio.
Hint: Even $(2 - \frac{1}{n^n})$–competitive is better than 2–competitive!

c) How many moves does the following strategy need in expectation where the sole player has no memory: In each move, turn over two different cards uniformly at random. If they form a pair, collect the pair, else turn the cards over again. Repeat until all pairs are collected.