Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Distributed Computing

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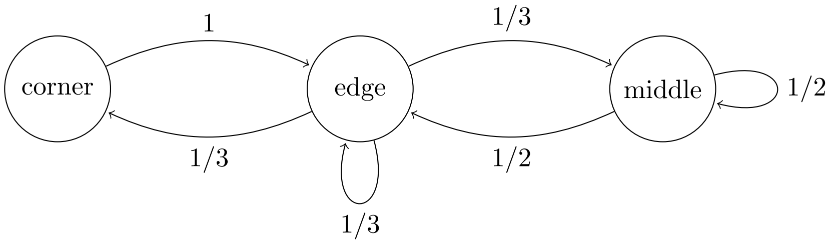
Discrete Event Systems

Solution to Exercise Sheet 7

1 Night Watch

a) Observe that the problem is symmetric, e.g., from all four corners, the situation looks the same, and the probability of being in a specific corner room is the same for all corners. The same holds for rooms at the border and for rooms in the middle. Thus, instead of using 16 states, we consider the following simplified Markov chain consisting of three states only:



The transition matrix M is given as follows.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

To calculate the steady state probability, we have to calculate the eigenvector of M to the eigenvalue 1, that is solve the equation

$$v \cdot M = v$$

for v (Be careful to multiply M from the right side). Intuitively this means that if we have a state distribution v and applying the transition matrix does not change this distribution v, then v is the steady state distribution.

For $v = (c, e, m)$, we get the following system of linear equations from the above equation.

$$c = \frac{1}{3}e \qquad e = \frac{1}{3}e + \frac{1}{2}m + c \qquad 1 = c + e + m$$

Solving this equation system gives: $c = 1/6$. The probability of being in a specific corner is therefore $1/6 \cdot 1/4 = 1/24$.

b) Since the two walks are independent, we have – according to the inclusion-exclusion principle (Einschluss-Ausschluss-Verfahren) –

$$\frac{1}{24} + \frac{1}{24} - \left(\frac{1}{24}\right)^2 \approx 0.082 \ .$$

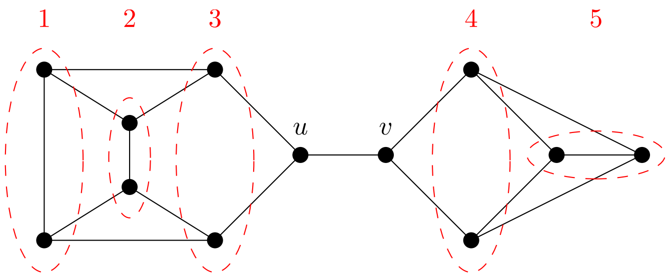
2 Hitting Time of a Simple Random Walk

a) Consider the line graph of three nodes u, v, w with edges $\{u, v\}$ and $\{v, w\}$. In this graph $h_{uv} = 1$, but $h_{vu} > 1$.

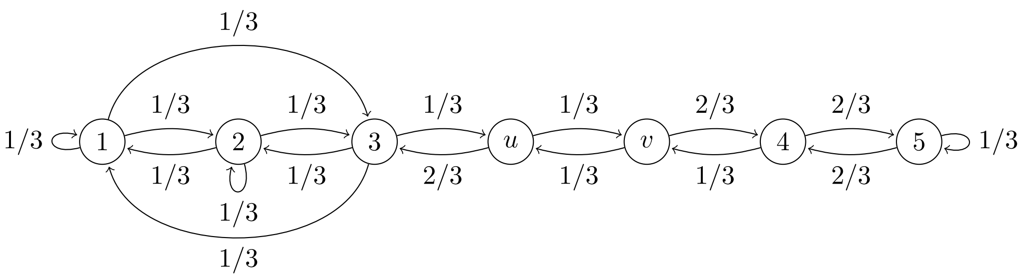
b) Two simple families of graphs that satisfy $h_{uv} = h_{vu}$ due to their inherent symmetry are ring graphs and complete graphs.

c) Intuitively speaking, if the graph "looks the same" from the view of two nodes u and v, then the hitting time will satisfy $h_{uv} = h_{vu}$. To put it more formally, the hitting time for two nodes is symmetric, if there exists a graph automorphism[1] $\phi : V(G) \to V(G)$ which satisfies $\phi(v) = u$ and $\phi(u) = v$.

It is important to notice that it is not sufficient that the graph is regular. To understand this, consider the following 3-regular graph and the edge (u, v).



To calculate the hitting times h_{uv} and h_{vu}, we first convert it to a simplied Markov chain by merging equivalent states as indicated by the dashed ellipses and name them according to the numbers above. We obtain the following Markov chain.



After observing that the hitting times h_{uv} and h_{vu} in this simplied Markov chain are identical to the ones in the original graph, we calculate $h_{uv} = 21$ and $h_{vu} = 15$. An intuitive argument why it takes longer to get from u to v than vice versa is that if the walker takes a wrong turn, it spends more time in the larger graph on the left than in the smaller one on the right.

You might wonder why one needs such a large graph to show a rather trivial argument. However, this is the smallest graph that we could come up with. If you know one with fewer nodes/edges, we would be happy to learn about it.

3 The Knight and the Bunny

a) As noted in the exercise, we create a Markov chain MC with n^2 states of the form (k, b) for all possible choices of k and b from V. Let $N(k)$ be the set of neighbors of k in G and $N(b)$ be the set of neighbors of b in G. Then a node (k, b) is connected to $|N(k)| \cdot |N(b)|$ nodes in the Markov chain. What is the number of edges $E(MC)$ in MC? It holds that

[1] An automorphism ϕ of a graph G is a permutation of the graph's vertices that "respects the structure" of G, that is, if $\{u, v\}$ is an edge in G, then $\{\phi(u), \phi(v)\}$ is also an edge.

the sum of all neighbors of all nodes in MC (note: We count each edge twice now!) can be expressed as (see the definition above):

$$2|E(MC)| = \sum_i \sum_j |N((i,j))| = \sum_i \sum_j |N(i)| \cdot |N(j)| \tag{1}$$

b) We can now take this finite sum apart for the following insight of $2|E(MC)| = 4m^2$:

$$\sum_i \sum_j |N(i)| \cdot |N(j)| = \sum_i |N(i)| \cdot \sum_j |N(j)| = 2m \cdot 2m = 4m^2 \tag{2}$$

Let $u = (i_u, j_u)$ and $v = (i_v, j_v)$ be nodes in MC. According to the lecture (see the slide "Cover Time von Random Walks"), it holds that $h_{uv} \leq 2|E(MC)|$ if the edge (u,v) exists. Therefore

$$h_{uv} \leq 2|E(MC)| \leq 4m^2 \ . \tag{3}$$

c) To show the upper bound, we now need to show that for any node (i,j) there exists a path of length at most $3n$ to a node where the Bunny kills the Knight, i.e., a node of the form (v,v). We now show that such a path exists between any (i,j) and the corresponding (j,j):

- The Bunny at j can always go back to the node j in two steps (the graph is undirected!)
- Since the graph G is connected, there is a path P of length $p < n$ from i to j in G.
  - If p is even, then the Knight will run into the Bunny on node j.
  - If p is odd, then the Knight will miss the Bunny on node j.
  - But the graph contains an odd cycle, meaning that if p is odd, then the Knight could travel to this cycle, use it, and then go to node j - which makes the number of edges traversed even (and the Knight could meet his fate)! We can bound the number of traversed edges from above with $3n$ in this case.
- Each edge on the path requires at most $4m^2$ steps, giving therefore the desired upper bound of $12 \cdot n \cdot m^2$.