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Distributed Computing

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Discrete Event Systems

Solution to Exercise Sheet 8

1 Colour Blindness/Daltonism

Since the sample size n is large and the probability for someone being colour blind is small, we can estimate the distribution of colour blind people with the Poisson distribution.

The Poisson distribution

The Poisson distribution is a *discrete* probability distribution which is applied often to approximate the binomial distribution for large number n of repetitions and small success probability p of the underlying Bernoulli experiments. Usually, it is used to model situations where stochastical events happen with a given rate in a given quantity. If we know that an event on average happens once within the quantity q_1 and we are interested in the number of events X in another quantity q_2, then X is Poisson-distributed with parameter $\lambda = q_2/q_1$.

$$\Pr[X = x] = \frac{\lambda^x}{x!} e^{-\lambda}$$

a) The average rate of colour blind people is 2 out of 100 or one blind person in 50, hence we have $q_1 = 50$. We are interested in the number X of colour blind persons in a sample of 100 persons, hence $q_2 = 100$ and $\lambda = q_2/q_1 = 2$. Then the probability that x persons out of 100 are colour blind is given by

$$\Pr[X = x] = e^{-2} \cdot \frac{2^x}{x!} \quad .$$

The probability that at most three persons out of 100 are colour blind is given by

$$\begin{aligned}
\Pr[X \le 3] &= \Pr[X = 3] + \Pr[X = 2] + \Pr[X = 1] + \Pr[X = 0] \\
&= e^{-2} \cdot \frac{2^3}{3!} + e^{-2} \cdot \frac{2^2}{2!} + e^{-2} \cdot \frac{2^1}{1!} + e^{-2} \cdot \frac{2^0}{0!} \\
&= e^{-2} \cdot \left(\frac{8}{6} + \frac{4}{2} + \frac{2}{1} + 1 \right) \\
&= \frac{19}{3} e^{-2} \\
&\approx 0.857
\end{aligned}$$

b) Now we are interested in the sample size n such that at least one person is colour blind with probability 90%, i.e. $q_2 = n$ and $\lambda = q_1/q_2 = n/50$. The probability that at least one person is colour blind in a sample of size n is now given by

$$\begin{aligned}\Pr[X \geq 1] &= 1 - \Pr[X = 0] \\ &= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} \\ &= 1 - e^{-n/50} \quad .\end{aligned}$$

Setting $\Pr[X \geq 1] \geq 90\%$ and solving this inequality for n yields $n \geq 116$. Hence, in a sample of 116 persons we have at least one colour blind person with probability 90%.

2 Gloriabar

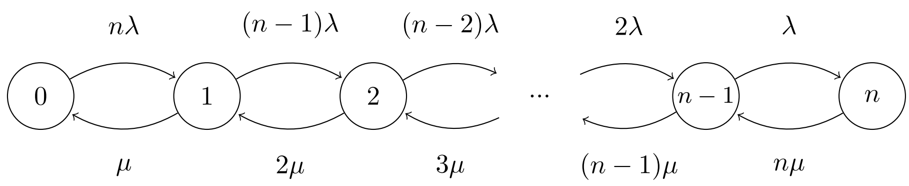
a) The situation can be modeled by a M/M/1 queue. We have an arrival rate of $\lambda = 540/(90 \cdot 60) = 1/10$ (persons per second), and $\mu = 1/9$ (persons per second). Thus $\rho = \lambda/\mu = 9/10$. We can apply Little's Law (slides 76 ff.) and therefore, we can use the formulae for the response and waiting time from slide 79: The expected waiting time is $W = \rho/(\mu - \lambda) = 81$ seconds. The expected time until the student has paid for her menu is given by $T = 1/(\mu - \lambda) = 90$ seconds.

b) We use the formula for the expected number of jobs in the queue from slide 79 and obtain queue length of $N_Q = \rho^2/(1-\rho) = 8.1$.

c) We require that $T = 1/(\mu - 0.1) = 90/2$. Thus, $\mu = 11/90$, i.e., instead of 9 secs, the service time should be $90/11 \approx 8.2$ secs.

3 Beachvolleyball

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let π_i be the probability of state i in the equilibrium. From slide 87, we know that

$$\pi_i = \pi_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}$$

and thus

$$\pi_i = \pi_0 \cdot \frac{\lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_i}.$$

Applying this formula to our process yields

$$\pi_i = \pi_0 \cdot \frac{n(n-1) \cdot \dots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \dots \cdot i \cdot \mu^i} = \pi_0 \cdot \binom{n}{i} \cdot \rho^i \tag{1}$$

where $\rho := \frac{\lambda}{\mu}$. We know that the sum of all probabilities equals 1, so we have

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{n}{i} \rho^i = 1$$

Using the given formula for the binomial series

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$$

we obtain

$$\pi_0 (1+\rho)^n = 1 \ .$$

Finally, we obtain

$$\pi_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n} \ .$$

c) (i) It is $\rho = 3/9 = 1/3$. We calculate the probability that there are less than two fit players:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \left(\frac{3}{4}\right)^5 \cdot \left(1 + \frac{5}{3}\right) \\
&= \frac{3^5}{2^{10}} \cdot \frac{8}{3} \\
&= \frac{3^4}{2^7} \approx 0.63
\end{aligned}$$

Thus, the Disco team cannot participate in the tournament with probability 0.63.

(ii) Now, $\rho = 4/2 = 2$. Again, we calculate $\pi_0 + \pi_1$:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \frac{1}{3^5} \cdot (1 + 2 \cdot 5) \\
&= \frac{11}{3^5} \approx 0.045
\end{aligned}$$

Hence, the probability that the Disco team cannot participate is only 0.045!

(iii) In general, if $\rho \geq 1$, an M/M/1 queue might grow infinitely and therefore doesn't have a stationary distribution. This cannot happen in this birth-and-death process, though, because there is only a bounded number of states. Hence, the process has a stationary distribution even for $\rho \geq 1$.

4 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. According to slide 90, there is an equilibrium iff

$$\rho = \lambda/(2\mu) < 1 \ .$$

For the stationary distribution, it holds that

$$
\begin{aligned}
\pi_0 &= \frac{1}{\left(\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!}\right) + \frac{(\rho m)^m}{m!(1-\rho)}} \\
&= \frac{1}{\frac{(2\rho)^0}{0!} + \frac{(2\rho)^1}{1!} + \frac{(2\rho)^2}{2!(1-\rho)}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{(2(1-\rho))}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{2(1-\rho)}} \\
&= \frac{1}{\frac{2(1-\rho)+4\rho(1-\rho)+4\rho^2}{2(1-\rho)}} \\
&= \frac{2(1-\rho)}{2 - 2\rho + 4\rho - 4\rho^2 + 4\rho^2} \\
&= \frac{2(1-\rho)}{2 + 2\rho} \\
&= \frac{1-\rho}{1+\rho} \ .
\end{aligned}
$$