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Discrete Event Systems

Solution to Exercise Sheet 6

1 An Unsolvable Problem

a) It is surprisingly easy to prove that your boss is demanding too much. Assume a function `halt(P: Program): boolean` which takes a program `P` as a parameter and returns a boolean value denoting whether `P` terminates or not.

Now consider the following program `X` which calls the `halt()` function with itself as an argument just to do the contrary:

```
function X() {
  if (halt(X))
    while(true);
  else
    return;
}
```

Obviously, if `halt(X)` is true `X` will loop forever, and vice versa.

b) If the simulation stops we can definitively decide that the program does not contain an endless loop. However, while the simulation is still running, we do not know whether it will finish in the next two seconds or run forever. Put differently: There is no upper bound on the execution time of the simulation after which we can be sure that the program contains an endless loop.

c) As we have seen, it is not possible to predict whether a general program terminates or not. However, under certain constraints we can solve the halting problem all the same. For example, consider a restricted language with only one form of a loop (no recursion etc.):

```
for (init; end; inc) {...}
```

where `init`, `end` and `inc` are constants in $\mathbb{Z}$. The loop starts with the value `init` and adds `inc` to `init` in every round until this sum exceeds `end` if `end` > 0 or until it falls below `end` if `end` < 0. Obviously, there is a simple way to decide whether a program written in this language terminates: For every loop, we check whether $\text{sgn}($`inc`$) = \text{sgn}($`end`$)$, where $\text{sgn}(\cdot)$ is the algebraic sign. If not, the program contains an endless loop (unless `init` itself already fulfills the termination criterion which is also easy to verify).

2 Dolce Vita in Rome

We define the following random variables.

$$X = \text{number of ice creams bought in total}$$
$$X_i = \text{indicator variable for buying ice cream at shop } i$$

That means X_i is 1 if Hector and Rachel buy ice cream at shop i and 0 otherwise. Since the probability that the i-th shop is the best so far equals $\frac{1}{i}$ and the expectation of an indicator variable is simply the probability of it being 1, we have

$$\mathbf{E}[X_i] = \frac{1}{i} \ .$$

Furthermore, we can express X as $\sum_{i=1}^{n} X_i$ and by using linearity of expectation, we obtain:

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} \frac{1}{i} = H_n \ .$$

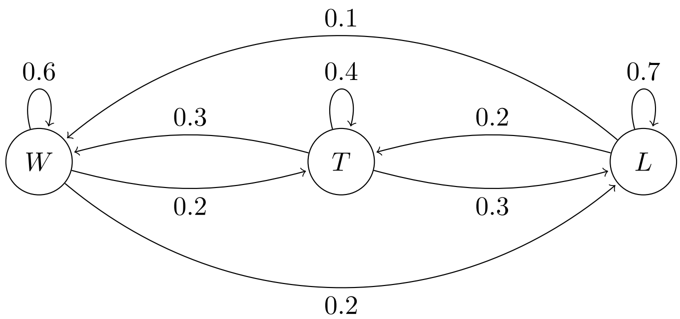
Here H_n is the n-th *Harmonic Number.* H_n grows about as fast as the natural logarithm of n. The reason for this is that the sum of the first n harmonic numbers can be approximated by

$$\int_1^n \frac{1}{x} \mathrm{d}x = \ln(n) \ .$$

More precisely, we have $H_n = \ln(n) + \mathcal{O}(1)$ and thus the two students roughly consume a logarithmic number of ice creams (in the total number of shops n).

3 Soccer Betting

a) The following Markov chain models the different transition probabilities (W:Win, T:Tie, L:Loss):



b) The transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \ .$$

As you might have noticed, we gave redundant information here. You only need the information that the FCB lost its last game. Thus, the Markov chain is currently in the state L and hence, the initial vector is $q_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. The probability distribution q_2 for the game against the FC Zurich is therefore given by

$$\begin{aligned} q_2 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P &= \begin{pmatrix} 0.1 & 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 0.19 & 0.24 & 0.57 \end{pmatrix} \ . \end{aligned}$$

(Note that q_0 must be a row vector, not a column vector.)

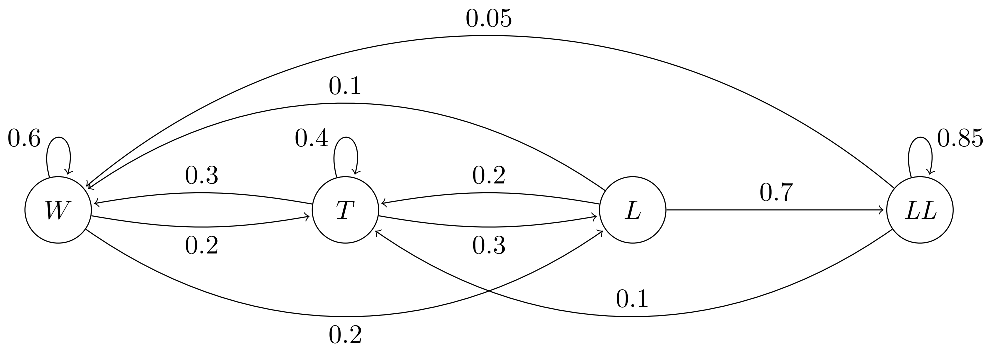
Hint: We exploited the associativity of the matrix multiplication to avoid having to calculate P^2 explicitly. This is usually a good "trick" to avoid extensive and error-prone calculations if no calculator is at hand (as for example in an exam situation $\ddot\smile$).

Given the quotas of the exercise, the expected profit for each of the three possibilities (W, T, L) calculates as follows.

$$\begin{aligned}\mathbf{E}[W] &= 0.19 \cdot 3.5 = 0.665\\ \mathbf{E}[T] &= 0.24 \cdot 4 \quad = 0.96\\ \mathbf{E}[L] &= 0.57 \cdot 1.5 = 0.855\end{aligned}$$

Therefore, the best choice is not to bet at all since the expected profit is smaller than 1 for every choice. If a "sales representative" of the Swiss gambling mafia were to force you to bet, you would be best off with betting on a tie, though.

c) The new Markov chain model looks like this. In addition to the three states W, T, and L, there is now a new state LL which is reached if the team has lost twice in a row.



The new transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} . \tag{1}$$

As the FCB has and lost its last two games, the Markov chain is currently in the state $q_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. The probabilities for the game against the FC Zurich can again be computed as follows.

$$\begin{aligned} q_3 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P &= \begin{pmatrix} 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \\ &= \begin{pmatrix} 0.1025 & 0.135 & 0.04 & 0.7225 \end{pmatrix} \end{aligned}$$

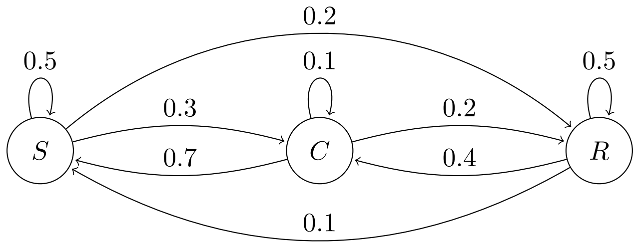
Finally, we can compute the expected profit for each of the three possible bets:

$$\begin{aligned}\mathbf{E}[W] &= 0.1025 \cdot 3.5 &&= 0.35875\\ \mathbf{E}[T] &= 0.135 \cdot 4 &&= 0.54\\ \mathbf{E}[L] &= (0.04 + 0.7225) \cdot 1.5 &&= 1.14375\ .\end{aligned}$$

Now, the best choice is to bet on a loss. Clearly, the addition of the state LL worsens the situation for FCB.

4 The Winter Coat Problem

a) The following Markov chain models the weather situation of Robinson's island.



b) We need to determine the expected hitting time h_{SS}. Using the formula of slide 35, we obtain the following equation system:

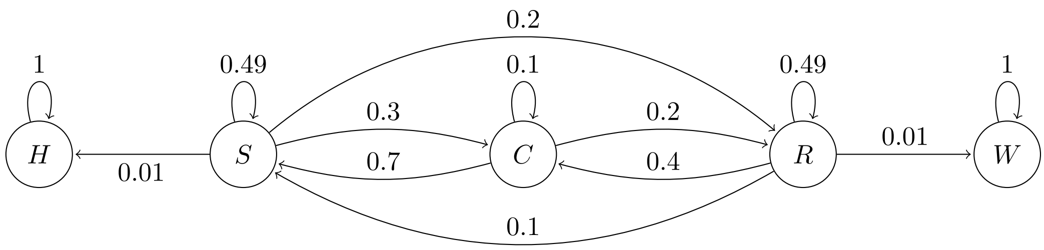
$$h_{SS} = 1 + 0.3h_{CS} + 0.2h_{RS} \tag{2}$$
$$h_{CS} = 1 + 0.1h_{CS} + 0.2h_{RS} \tag{3}$$
$$h_{RS} = 1 + 0.4h_{CS} + 0.5h_{RS} \tag{4}$$

(2) and (3) yield that $h_{CS} = \frac{5}{6}h_{SS}$, from (2) and (4) we obtain that $h_{RS} = \frac{40}{23}h_{SS} - \frac{10}{23}$. Plugging these results into (2), we obtain

$$h_{SS} = 1 + 0.3\left(\frac{5}{6}h_{SS}\right) + 0.2\left(\frac{40}{23}h_{SS} - \frac{10}{23}\right)$$
$$\Leftrightarrow \quad h_{SS} = \frac{1 - \frac{2}{23}}{1 - \frac{1}{4} - \frac{8}{23}} = \frac{84}{37} \approx 2.27$$

Thus, Mr. Robinson has to wait 2.27 days (in expectation) until having again a sunny day.

c) The modified Markov chain looks as following:



d) We need to determine the arrival probability f_{SW}, the probability that the weather will turn to winter. Using the formula of slide 35, we obtain the following equation system:

$$f_{SW} = 0 + 0.3f_{CW} + 0.2f_{RW} + 0.49f_{SW} + 0.01f_{HW} \tag{5}$$
$$f_{CW} = 0 + 0.7f_{SW} + 0.2f_{RW} + 0.1f_{CW} \tag{6}$$
$$f_{RW} = 0.01 + 0.4f_{CW} + 0.1f_{SW} + 0.49f_{RW} \tag{7}$$
$$f_{HW} = 0 \tag{8}$$

Solving the equation system yields

$$f_{SW} = \frac{240}{619},\ f_{RW} = \frac{249}{619},\ f_{CW} = \frac{242}{619}$$

And therefore, the probability that the weather turns to winter (snowing) and Mr. Robinson needs a winter coat is $\frac{240}{619} \approx 0.39$. Note that $f_{SH} = 1 - f_{SW} = \frac{379}{619}$ because all state sequences that do not end up in state W eventually end up in state H.