Chapter 3

Topology Control

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Theorem 3.1. *$MST \subseteq RNG \subseteq GG \subseteq DT$.*

Proof. To simplify the proof, assume that the points are in arbitrary position, that is, that there are not two pairs of points with exactly the same distance.

1. MST $\subseteq$ RNG: For the sake of contradiction, assume that there is an edge e with $e \in MST$ and $e \notin RNG$. If edge $e = (u, v)$ is not in RNG, the lune of (u, v) cannot be empty, i.e. there is a node w in the lune of (u, v), i.e. $|u, w| < |e|$ and $|v, w| < |e|$. When removing e, the MST gets partitioned into two subtrees, MST_u and MST_v. Without loss of generality assume that w belongs to MST_u. Now we add the edge (w, v) to the two subtrees, again having a connected spanning tree, however, because $|v, w| < |e|$ with lower weight than the original MST. This is a contradiction.

2. RNG $\subseteq$ GG: This is because the circle of the GG is contained in the lune of the RNG. In other words, if the circle is not empty (i.e., edge not in GG), then the lune is not empty (edge not in RNG).

3. GG $\subseteq$ DT: The DT has two popular definitions: i) three nodes are pairwise connected if there is a circle spanning the three nodes (but no others), and ii) two nodes are connected if there is a circle that only contains the two nodes. Definition ii) is a generalization of the GG definition.

□

Remarks:

- In the GG, a the border of the circle is considered part of the circle. In the RNG, the border of the lune is not considered part of the lune.

Theorem 3.2. *MST, RNG, GG, and DT are i) connected and ii) planar.*

Proof. Connectivity follows because the MST is (by definition) connected, and contained in all the other graphs. For planarity, is suffices to show that DT is planar, because all other graphs are contained in DT.

Again, to simplify the proof, assume that the points are in arbitrary position, that is, there are no four points on any circle. For the sake of contradiction, assume that two edges (u, v) and (x, y) are intersecting. Since there are no four nodes (u, v, x, y) on a circle, we look at the four possible circles. In at least one case, one of the four nodes is inside the circle of the other three,[1] contradicting the first definition of the DT, hence the edges cannot exist.

The proof that the RNG is planar is less geometric (in other words, better suited for this class). Again we take the two intersecting edges (u, v) and (x, y), and study the tetragon (u, x, v, y). The angles in a tetragon sum up to 360 degrees, hence one of the four corner angles will be at least 90 degrees. Without loss of generality, let x be that node. Using Thales' law, x is within (including the border) the GG circle of edge (u, v), hence (u, v) does not exist in the GG. Contradiction. □

Theorem 3.3. *The GG contains the so-called minimum energy path, where the energy of an edge e is often denoted as $E(e) = |e|^\alpha$ for some path-loss exponent $\alpha \geq 2$, and the energy of a path P (denoted as $E(P)$) is the sum of the energies of its edges.*

Proof. Let P be a minimum energy path in the original graph. Assume that there is an edge $e \in P$ which is not in GG. Then there is a node w in the GG circle of edge $e = (u, v)$. Now let path P' be path P, without edge e, in addition with edges (u, w) and (w, v). The energy of P' is $E(P') = E(P) - E(e) + E(u, w) + E(w, v) = E(P) - |e|^\alpha + |u, w|^\alpha + |w, v|^\alpha$. Since the angle (u, w, v) is more than 90 degrees, using Pythagoras for $\alpha \geq 2$ we know that $|e|^\alpha > |u, w|^\alpha + |w, v|^\alpha$. In other words, $E(P') < E(P)$ which is a contradiction. □

[1] Well, this seems intuitive, but obviously needs a proof that we omit for space reasons.