# Discrete Event Systems – lecture summary

Andreas Müller

March 6, 2007

**Abstract**

This summary is based on the slides Discrete Event Systems by Christoph Stamm, WS06/07.

## Contents

# 1 Automata and Languages

## 1.1 Definitions

**Alphabet** A set of symbols (characters, letters).

**String** A string or word is a sequence of symbols of an →alphabet.

- The empty string contains no symbols and is denoted by $\varepsilon$ (don't confuse with empty set $\emptyset$, which is not a string, but rather a set of no strings).
- The length of the string is the number of symbols, e.g. $|\varepsilon| = 0$.

**Regular Language** A language $L$ is called a regular language, if there is a FA, that accepts the language $L$. All finite languages are regular.

**Infinite Language** A language is infinite if it contains an infinite number of strings. A language can be infinite but regular (example: The language that consists of binary strings with an odd number of ones).

## 1.2 Regular Operations

- Union ($\bigcup$): Match one of the patterns
- Concatenation ($\bullet$): Match patterns in sequence
- Kleene-star ($*$): Match pattern 0 or more times
- Kleene-plus ($+$): Match pattern 1 or more times

The result of a regular operator applied to a regular language is always a regular language. In other words, regular languages are closed under the operations of union, concatenation and Kleene-star/plus.

## 1.3 Finite Automata (FA) [1/11]

### 1.3.1 Definition: Finite Automaton

A finite automaton (FA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- $Q$ is a finite set called the states
- $\Sigma$ is a finite set called the alphabet
- $\delta : Q \times \Sigma \to Q$ is the transformation function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states (final states)

### 1.3.2 Definition: Accepted Language

The language accepted (or recognized) by a finite automaton $M$ is the set of all strings accepted by $M$. It is denoted by $L(M)$.

- Not all languages can be described as the accepted language of an FA.

## 1.4 Regular Expressions (REX) [1/22]

### 1.4.1 Regular Expressions in UNIX

Regular Operations:

| Operation | Symbol | UNIX version | Meaning |
|---|---|---|---|
| Union | $\bigcup$ | \| | match one of the patterns |
| Concatenation | $\bullet$ | implicit | match patterns in sequence |
| Kleene-star | $*$ | $*$ | match pattern 0 or more times |
| Kleene-plus | $+$ | $+$ | match pattern 1 or more times |

Match between 10 and 15 times: $\{10, 15\}$
Operator precedence is $*, \bullet, \bigcup$

### Command Line:

- `egrep -i 'expression'`
  - -i: Ignore Case
- `perl -wlne'print if /expression/' $file`
- UNIX searches for lines *containing* the given string, to search for lines consisting of the string:
  - `egrep '^expression$'`

## 1.5 Non-deterministic Finite Automata (NFA) [1/48]

An NFA is an automaton whose transitions don't need to be deterministic, i.e. there can be multiple transitions from a state for the same symbol. If the automaton encounters such a transition, it is assumed to be in all possible states simultaneously. Any labeled graph with a start state is an NFA.

**Definition:** A non-deterministic finite automaton (NFA) is encapsulated by $M = (Q, \Sigma, \delta, q_0, F)$ in the same way as an FA, except that $\delta$ has different domain and co-domain: $\delta : Q \times \sigma_\varepsilon \to P(Q)$, where $P(Q)$ is the power set of $Q$, so that $\delta(q, a)$ is the set of all endpoints of edges from $q$ which are labeled by $a$.

**Definition:** A string $u$ is accepted by an NFA $M$ iff there exists a path starting at $q_0$ which is labeled by $u$ and ends in an accept state. The language accepted by $M$ is the set of all strings accepted by $M$ and is denoted by $L(M)$.

### 1.5.1 Regular Operations for NFA

NFA are closed under regular operations. Regular operations can be applied easily to NFA:

**Union ($A \bigcup B$):** Put $A$ and $B$ in parallel. Create new start state and conect it to all former start states with $\varepsilon$-edges.

**Concatenation ($A \bullet B$):** Use start states from $A$, connect all accept states of $A$ to all start states from $B$ via $\varepsilon$-edges and use accept states from $B$ as new accept states.

**Kleene-plus ($A+$):** Loop back all accept states to start states via $\varepsilon$-edges.

**Kleene-star ($A*$):** Same as Kleene-plus, but create a new, accepting start state that links to the old start state with an $\epsilon$edge.

## 1.6 State Minimization

- omit non-reachable states
- find equivalent states and join them

## 1.7 Transitions between FA, NFA and REX

FA, NFA and REX are equivalent.

### 1.7.1 NFA→FA

- always minimize the automaton first
- create power set of states ($n$ states $\to 2^n$ states)
- each new state that contains an accept state becomes an accept state
- new start state: set of old start state and all states that ca recursively reached from there with $\varepsilon$

### 1.7.2 REX→NFA [1/69]

→ slide 1/69

### 1.7.3 NFA→REX [1/83ff]

Via generic nondeterministic finite automaton (GNFA):

- a GNFA is an automaton whose edges are labeled by regular expressions with
  - a unique start state with in-degree 0 (but arrows to every other state)
  - a unique accept state (not the start state) with out-degree 0 (but arrows from every other state)
  - (an arrow from any state to any other state)

#### Conversion algorithm

1. construct GNFA from NFA
   (a) if there is more than one arrow from one state to another, unify them using $\bigcup$
   (b) create an unique start state with in-degree 0
   (c) create an unique accept state with out-degree 0
   (d) (if there is no arrow from one state to another, insert one with label $\emptyset$)
2. loop: as long as GNFA has more than 2 states, rip out arbitrary interior state and update labels
3. the last edge is labeled with the regular expression

## 1.8 Pumping Lemma [1/94]

For every regular language $L$, there is a number $p$ (pumping numer), such that any string in $L$ of legth $\geq p$ is pumpable within its first $p$ letters.

In other words, for all $u \in L$ with $|u| \geq p$, we can write:

- $u = xyz$ ($x$ is a prefix, $y$ is a suffix)
- $|y| \geq 1$ (mid-portion is non-empty)
- $|xy| \leq p$ (pumping occurs in first $p$ letters)
- $xy^iz \in L$ for all $i \geq 0$ (can pump $y$-portion)

(the pumping lemma *may* be able to prove that a language is non-regular, but not that a language is regular)

# 2 Smarter Automata

## 2.1 Context free grammars (CFG) [2/5]

**Definition:** A context free grammar consists of $(V, \Sigma, R, S)$ with

- $V$: a finite set of variables (or symbols, or non-terminals)
- $\Sigma$: a finite set of terminals (or the alphabet)

- $R$: a finite set of rules (or productions) of the form $v \to w$ with $v \in V$ an $w \in (\Sigma_\varepsilon \cup V)^*$
- $S \in V$: the start symbol

### 2.1.1 Derivation tree (parse tree) [2/10]

- each node is a symbol
  - root is the start symbol
  - parents are variables (non-terminals)
  - leaves are terminals
- leaves spell out the derived tree from left to right

### 2.1.2 Ambiguity

**Definition:** A string $x$ is said to be ambiguous relative the grammar $G$ if there are two essentially different ways to derive $x$ in $G$. I.e. $x$ admits two or more different parse trees (or $x$ admits different left-most or right-most derivations). A grammar $G$ is ambiguous if it $L(G)$ contains an ambiguous string.

### 2.1.3 Chomsky Normal Form [2/36]

We want only rules of the following forms:

- $S \to \varepsilon$ (only start state may produce $\varepsilon$)
- $A \to BC$ (dyadic variable productions)
- $A \to a$ (unit terminal productions)

Conversion:

1. Ensure that start variable doesn't appear on the right hand side of any rule (if it does, exchange it by the productions of $S$).
2. Remove all epsilon productions except from start variable (e.g.: $A \to \varepsilon|a|b|AB$ becomes $A \to a|b|AB|B$ etc)
3. Remove all unit variable productions of the form $A \to B$.
4. Add new variables and dyadic (2) variable rules to replace non-dyadic or non-variable productions.

## 2.2 Pushdown Automata (PDA) [2/21]

**Definition:** A pushdown automaton is a 6-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ with

- $Q$ (states), $\Sigma$ (input alphabet), $q_0$ (start state), $F$ (accept states) are the same as for FA
- $\Gamma$ is the stack alphabet
- $\delta$, for a given state, input letter and stack letter gives an output letter and a stack replacement:

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to P(Q \times \Gamma_\varepsilon)$$

**Labeling convention:**

$$\text{(p)} \xrightarrow{x,y \to z} \text{(q)}$$

If at state p with input $x$ and stack $y$, go to state q and replace $y$ on stack with $z$.

- $x = \varepsilon$: ignore input, don't read
- $y = \varepsilon$: ignore top of stack and push z
- $z = \varepsilon$: pop y

## 2.3 Tandem Pumping [2/46]

Tandem Pumping can prove that a language is not a context free grammer. To prove that a language is a CFG, a CFG that creates the language or an automaton (DFA, NFA or REX) that accepts it can be given.

## 2.4 Transducer [2/50]

- A finite state transducer (FST) is a type of finite automaton whose output is a string instead of accept or reject.
- Each transition is labeled with two symbols $a/b$ where $a$ is the input symbol (as for automata) and $b$ the output symbol.

## 2.5 Turing Machine [2/52]

**Definition** A Turing machine (TM) is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ with

- $Q$ (states), $\Sigma$ (input alphabet), $q_0$ (start state) are the same as for FA
- $q_{acc}$ and $q_{rej}$ are accept and reject states
- $\Gamma$ is the tape aphabet, which necessarily contains the blank symbol $\square$ and the input alphabet $\Sigma$
- $\delta$ is the transition

  $$\delta : \ (Q - \{q_{acc}, q_{rej}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}\}$$

  where $L$, $R$ is a left or right shift on the tape

A string $s$ is accepted if the TM with $s$ on the tape and the head on the left most state eventually enters the accept state.

**Labeling convention:**

$$\text{(p)} \xrightarrow{x \to y|R} \text{(q)}$$

If at state p with band input $x$, write $y$ to band and go right on the tape and to state q.

$$\text{(p)} \xrightarrow{\square|L} \text{(q)}$$

If at state p with band input $\square$ (empty), leave $\square$ on band and go left on the tape and to state q.

# 3 Specification Models

## 3.1 State Charts [3/2]

State charts are an extension of automata:

- notation:

$$A \xrightarrow{event[condition]/reaction} B$$

  - event: can be either internally or externally generated; logical function of multiple events is allowed
  - condition: refers to values or variables that keep their value until they are reassigned; multiple conditions possible
  - the transition is enabled if event *and* condition is true
  - reaction: assignment to variables and/or creation of events
- Superstates
  - OR-superstates → exactly one of the substates is active, when the superstate is active
  - AND-superstates → all substates are active, when the superstate is active

## 3.2 Petri Nets [3/22]

**Definition:** A Petri net is a bipartite, directed graph defined by a tuple $(S, T, F, M_0)$, where

- $S$ is a set of places $p_i$
- $T$ is a set of transitions $t_i$
- $F$ is a set of edges (flow relations) $f_i$
- $M_0 : S \to N$ is the initial marking

A transition may fire if all incoming egdes have the possibility to consume a token. All places at outgoing edges will then get a token. Which transition will fire is non-deterministic.

### 3.2.1 Analysis

**Properties of petri nets [3/38]:**

**Reachability** A marking $M_n$ is reachable iff there exists a sequence of firings $\{t_1, t_2, ..., t_n\}$ so that $M_n = M_0 \cdot t_1 \cdot t_2 \cdot \ldots t_n$

**$K$-Boundedness** A petri net is $K$-bounded if the number of tokens in every place never exceeds $K$

**Safety** 1-boundedness: Every node holds at most 1 token at any time

**Liveness** starting from the current state, can we eventually fire any transition?

**Properties of transitions [3/39]:** A transition $t$ in a Petri net $(N, M_0)$ is

**dead** if $t$ cannot be fired in any firing sequence of $L(M_0)$

**L1-live** if $t$ can be fired at least once in some sequence of $L(M_0)$

**L2-live** if, $\forall k \in \mathbb{N}^+$, $t$ can be fired at least $k$ times in some sequence of $L(M_0)$

**L3-live** if $t$ apears infinitely often in some infinite sequence of $L(M_0)$

**L4-live** (live) if $t$ is L1-live for every marking reachable from $M_0$

**Coverability Tree [3/42]:** The coverability tree is a tree of all reachable token distributions. To denote an arbitrary number of tokens, a special symbol $\omega$ is used.

Algorithm to create the tree:

- label initial marking $M_0$ as root and tag it as *new*
- **while** *new* markings exist, pick one, say $M$ and **do**
  - if $M$ is identical to a marking on the way from the root to $M$, mark it as *old*; **continue**
  - if no transitions are enabled at $M$, tag it as *deadend*
  - **for each** enabled transition $t$ at $M$ **do**
    - obtain marking $M' = M \cdot t$
    - if there exists a marking $M''$ on the way from the root to $M$ s.t. $M'(p) \geq M''(p)$ for each place $p$ and $M' \neq M''$, replace $M'(p)$ with $\omega$ for $p$ where $M'(p) > M''(p)$
    - introduce $M'$ as a node, draw an arrow with label $t$ from $M$ to $M'$ and tag $M'$ as new

Results from the Coverability Tree $T$:

- the net is **bounded** iff $\omega$ does not appear in any node label
- the net is safe if only 0 and 1 appear in the node labels of $T$
- a transition $t$ is **dead** iff it does not appear as edge in $T$
- if $M$ is reachable from $M_0$, then there exists a node $M'$ s.t. $M \leq M'$ (necessary but not sufficient)
- for *bounded* petri nets, this tree is also called **reachability tree**; it contains all reachable markings
- **deadlocks** are possible if at lease one node is labeled with *deadend*

**Incidence Matrix:** The incidence matrix $A$ describes the tokenflow according for the different transitions:

- $A_{ij}$ = gain of tokens at node $i$ when transition $j$ fires
- in other words, $A$ is a matrix with a row for each place and and column for each transition; $-n$ denotes an outgoing transition of capacity $n$; $+n$ denotes an incoming transition of capacity $n$

- a marking $M$ is written as a column vector
- the firing vector $u_i$ describes the firing of transition $i$ and consists of all 0, except for the $i$-th position, where it has a 1
- a firing of transition $t_i$ at with a marking $M_k$ is written as

$$M_{k+1} = M_k + A \cdot u_i$$

Reachability:

- a marking $M_k$ is reachable from $M_0$ if there is a sequence of transitions $\{t_1, .., t_k\}$ such that $M_k = M_0 \cdot t_1 \cdot ... \cdot t_k$
- written with the incidence matrix:

$$M_k = M_0 + A \sum_{i=1}^{k} u_i$$

  this can be rewritten as

$$M_k - M_0 = \Delta M = A \cdot \vec{x}$$

- if $M_k$ is reachable, this equation must have a solution where all components of $\vec{x}$ are positive integers (necessary but not sufficient)

### 3.2.2 Extensions [3/52]

- weighted edges [3/30]
  - each edge $f_i$ has an associated weight $W(f_i)$
  - transition is active if each place $p_i$ connected through incoming edge $f_i$ contains at least $W(f_i)$ tokens
  - new tokens are generated based on weight of edges
  - still a regular petri net
- finite capacity [3/31]
  - each place $p_i$ can hold $K(p_i)$ tokens at most
  - to remove this constraint for $p$, add a complementary place $p'$, in way that the number of tokens in $p$ and $p'$ together is always $K(p)$
  - still a regular petri net
- colored petri nets: token carry values (colors)
  - can be transformed to regular petri net
- continuous petri net: token number can be real
  - can not be transformed to regular petri net
- inhibitor arcs: enable transition if a place contains no tokens
  - can not be transformed to regular petri net

# 4 Stochastic discrete event systems

## 4.1 Basics [4/5]

→ see other summaries

## 4.2 Stochastic processes in discrete time [4/16]

### 4.2.1 Markov-Chain

- next state only depends on current state, not on past states
- use probability matrices ($P$) for description:

$$q_{t+1} = q_t * P$$
$$q_t = q_0 * P^t$$

  (sanity check: row sums have to be 1)
- transition times (erwartete Übergangszeiten)

$$h_{ij} = 1 + \sum_{k:k\neq j} p_{ik} h_{kj}$$

  (if the expectation values $h_{ij}$ and $h_{kj}$ exist)
- arrival probabilities (Ankunftswahrscheinlichkeiten)

$$f_{ij} = p_{ij} + \sum_{k:k\neq j} p_{ik} f_{kj}$$

- stationary analysis [4/38]
  - bevaviour for $t \to \infty$
  - a vector $\pi$ with $\pi_j \geq 0$ and $\sum_{j\in S} \pi_j = 1$ is called a stationary distribution of the markov chain with transition matrix $P$, if $\pi = \pi \cdot P$
  - the stationary distribution $\pi$ is an Eigenvector of $P$ with Eigenvalue 1
  - TI
    - ⋆ `eigvl(`$P$`)`: eigenvalues
    - ⋆ `eigvc(`$P^T$`)`: eigenvectors in columns
    - ⋆ `eigvc(`$P^T$`)`$^T$`[2]`: second eigenvector as row vector
    - ⋆ `ans(1)/rownorm(ans(1))`: rescale to element sum=1

## 4.3 Stochastic processes in continuous time [4/50]

### 4.3.1 Distributions

(from WahrStat summary)

- **Gleichverteilung auf** $[a, b]$, $a < b$

- Dichte

$$f_{a,b}(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

- Verteilungsfunktion

$$F_{a,b}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b \end{cases}$$

- $E[X] = \frac{a+b}{2}$
- $\text{Var}[X] = \frac{(b-a)^2}{12}$

▶ **Normalverteilung** mit Parametern $\mu \in \mathbb{R}, \sigma > 0$:

- Dichte:

$$f_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x \in \mathbb{R}$$

- $E[X] = \mu$
- $\text{Var}[x] = \sigma^2$.
- Die Zufallsvariable $Z$ mit Verteilung $N(\mu, \sigma)$ (Normalverteilung mit Parametern $\mu,\ \sigma$) enstpricht der Zufallsvariablen $\frac{Z-\mu}{\sigma}$ mit Verteilung $N(0,1)$ (nützlich bei Verwendung einer Tabelle).

▶ **Exponentialverteilung** mit Parameter $\lambda > 0$:

- Dichte:

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{sonst} \end{cases}$$

- Verteilungsfunktion:

$$F_\lambda(x) = \int_{-\infty}^{x} f_\lambda(u)\mathrm{d}u = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

- $E[X] = \frac{1}{\lambda}$
- $\text{Var}[X] = \frac{1}{\lambda^2}$
- sind $X_1, .., X_n$ exponentialverteilt mit Parametern $\lambda_1, .., \lambda_n$, so ist das Minimum $\min(X_1, .., X_m)$ exponentialverteilt mit Parameter $\lambda = \lambda_1 + ... + \lambda_n$

### 4.3.2 Markov-chain in countinuous time [4/57]

▶ residence probabilites (?) (Aufenthaltswahrscheinlichkeiten) [4/62]

- start distribution $q(0)$: $q_i(0) = Pr[X(0) = i]$ for $i \in S$ ($S$: states)
- distribution at time $t$: $q_i(t) = Pr[X(t) = i]$ for $i \in S$
- change of residence probabilities:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} q_i(t)}_{\text{Aenderung}} = \underbrace{\sum_{j:j\neq i} q_j(t) \cdot v_{j,i}}_{\text{Zufluss}} - \underbrace{q_i(t) \cdot v_i}_{\text{Abfluss}}$$

    - not easy to solve
    - for a stationary distribution, $\frac{\mathrm{d}}{\mathrm{d}t} q_i(t) = 0$ must hold
    - for $t \to \infty$, we get a linear equation system, that must be solved by a stationary distribution $\pi$:

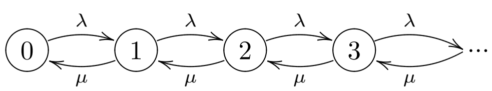
$$0 = \sum_{j:j \neq i} \pi_j \cdot v_{j,i} - \pi_i \cdot v_i$$

    - see [4/65] for an example with two states

### 4.3.3 Kendall-notation $X/Y/m$ for queues [4/68]

- $X$ stands for the distribution of the time between two incoming jobs
- $Y$ stands for the distribution of the job processing time (without waiting time)
- $m$ is the number of servers
- distributions are:
  - $D$: deterministic (fixed time)
  - $M$: memoryless (exponential distribution)
  - $G$: general (something else)

### 4.3.4 $M/M/1$-queues [4/70]

- arrival time distances and processing time exponentially distributed with parameter $\lambda$ resp. $\mu$
- traffic density: $\rho = \frac{\lambda}{\mu}$
- model with a Markov-chain in continuous time
  - states: number of jobs in system
  - state set: $S = \mathbb{N}_0$
  - transit rate from $i$ to $i+1$ is $\lambda$
  - transit rate from $i > 0$ to $i-1$ is $\mu$
- 

- stationary distribution [4/71]
  - for $\rho \geq 1$: no stationary solution, queue grows infinitely
  - for $\rho < 1$: stationary distribution $\pi$ with $\pi_k = (1-\rho)\rho^k$ for $k \geq 0$. Average server load (Auslastung): $1 - \pi_0 = \rho$
  - expected number of jobs in the system (queue+server):

$$N = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

- average response time:

$$T = \frac{N}{\lambda} = \frac{1}{\mu - \lambda}$$

- average waiting time without handling time:

$$W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda}$$

- average number of jobs in queue:

$$N_Q = \lambda W = \frac{\rho^2}{1 - \rho}$$

- generalisation of $M/M/1$-queue: see [4/87]

### 4.3.5 Little's Law [4/75]

Definitions:

- $N(t)$: number of jobs in the system (queue+server) at time $t$
- $\alpha(t)$: number of jobs that arrived in interval $[0, t]$
- $T_i$: response time for job $i$ (waiting+handling)

Average values at time $t$:

$$N_t = \frac{1}{t}\int_0^t N(\tau)\mathrm{d}\tau, \qquad \lambda_t = \frac{\alpha(t)}{t}, \qquad T_t = \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

Limit for $t \to \infty$:

$$N = \lim_{t\to\infty} N_t, \qquad \lambda = \lim_{t\to\infty} \lambda_t, \qquad T = \lim_{t\to\infty} T_t$$

If these limits exists and $\lim_{t\to\infty} \frac{\beta(t)}{t}$ exists and equals $\lambda$ ($\beta(t)$ is the number of finished jobs in interval $[0, t]$), then Littles Law tells us:

$$N = \lambda \cdot T$$

(this holds for strategies other than FCFS as well)

### 4.3.6 Time-Sharing

see[4/83]

### 4.3.7 M/M/1-queue with limited (to $n$) number of waiting spaces

- see [4/88]
- stationary distribution for waiting spaces:

$$\pi_0 = \frac{1}{\sum_{i=0}^{n} \rho^i} = \begin{cases} \frac{1}{n+1} & \text{for} \rho = 1 \\ \frac{1-\rho}{1-\rho^{n+1}} & \text{otherwise} \end{cases}$$

$$\pi_k = \rho^k \cdot \pi_0 \qquad \text{for } 1 \leq k \leq n$$

### 4.3.8 $M/M/m$-System

- see [4/90]
- equilibrium if $\rho = \dfrac{\lambda}{m\mu} < 1$
- stationary distribution for waiting spaces:

$$\pi_0 = \frac{1}{\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} + \frac{(\rho m)^m}{m!(1-\rho)}}$$

$$\pi_k = \begin{cases} \pi_0 \cdot \frac{\lambda^k}{\mu^k \cdot k!} = \pi_0 \cdot \frac{(\rho m)^k}{k!} & \text{for } 1 \leq k \leq m \\ \pi_0 \cdot \frac{\lambda^k}{\mu^k m! m^{k-m}} = \pi_0 \cdot \frac{\rho^k m^m}{m!} & \text{for } k \geq m \end{cases}$$

- probability $P_Q$ that an incoming job has to wait

$$P_Q = \frac{(\rho m)^m / (m!(1-p))}{\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} + \frac{(\rho m)^m}{m!(1-\rho)}} \qquad (\text{for } \rho = \frac{\lambda}{m\mu} < 1)$$

- expectation value for number of jobs in queue

$$N_Q = P_Q \cdot \frac{\rho}{1-\rho}$$

- average wait time

$$W = \frac{N_Q}{\lambda} = P_Q \cdot \frac{\rho}{\lambda(1-\rho)}$$

- average response time

$$T = W + \frac{1}{\mu} = \frac{P_Q}{m\mu - \lambda} + \frac{1}{\mu}$$

- average number of jobs in the system

$$N = \lambda T = \frac{\rho P_Q}{1-\rho} + m\rho$$

### 4.3.9 $M/M/m/m$-System

- see [4/94]
- $m$ servers, at most $m$ jobs
- stationary distribution

$$\pi_0 = \frac{1}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}$$

$$\pi_m = \frac{\left(\frac{\lambda}{\mu}\right)^m \frac{1}{m!}}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}$$

# 5 Worst-Case Event Systems [5]

## 5.1 Competitive Analysis

An online algorithm $A$ is called $c$-competitive if for all finite input sequences $I$ it holds that

$$\text{cost}_A(I) \leq c \cdot \text{cost}_{opt}(I) + k$$

where $k$ is an input-independant constant. If $k = 0$, $A$ is called strictly $c$-competitive.

## 5.2 Greedy Algorithm

A greedy will always choose the local optimum in the (most often false) hope to find the global optimum.