Discrete Event Systems
Solution to Exercise 6

1 An Unsolvable Problem

a) It is surprisingly easy to prove that your boss is demanding too much. Assume a function $halt(P : Program) : boolean$ which takes a program P as a parameter and returns a boolean value denoting whether P terminates or not.

Now consider the following program X which calls the *halt*() function with itself as an argument just to do the contrary:

```
function X(){
   if (halt(X))
     loop forever;
   else
     return;
}
```

Obviously, if $halt(X)$ is true X will loop forever, and vice versa.

b) If the simulation stops we can definitively decide that the program does not contain an endless loop. However, while the simulation is still running, we do not know whether it will finish in the next two seconds or run forever. Put differently: There is no upper bound on the execution time of the simulation after which we can be sure that the program contains an endless loop.

c) As we have seen, it is not possible to predict whether a general program terminates or not. However, under certain constraints we can solve the halting problem all the same. For example, consider a restricted language with only one form of a loop (no recursion etc.):

$$\texttt{for}(init, end, inc)\{...\}$$

where $init$, end and inc are constants in $\mathbb{Z}$. The loop starts with the value $init$ and adds inc to $init$ in every round until this sum exceeds end if $end > 0$ or until it falls below end if $end < 0$. Obviously, there is a simple way to decide whether a program written in this language terminates: For every loop, we check whether $sgn(inc) = sgn(end)$, where $sgn(\cdot)$ is the algebraic sign. If not, the program contains an endless loop (unless $init$ itself already fulfills the termination criterion which is also easy to verify).

2 Dolce Vita in Rome

Consider the following indicator variable for shop i: X_i is 1 if Hector and Rachel buy ice cream at the i-th shop and 0 otherwise. Since the probability that the i-th shop is the best so far equals $1/i$, we have $E[X_i] = 1/i \cdot 1 = 1/i$.

The total number of ice crams can be expressed by

$$X := X_1 + X_2 + ... + X_n.$$

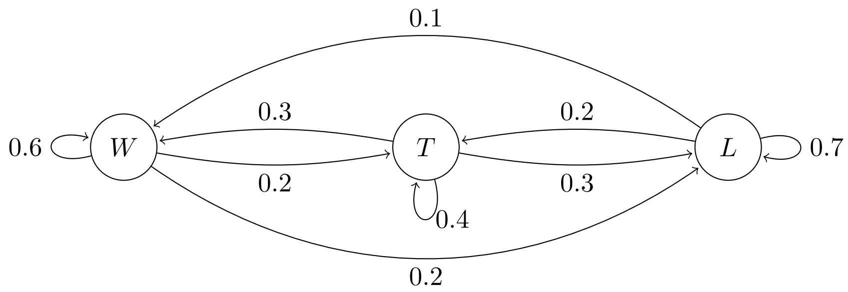
By using linearity of expectation, we obtain:

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = H_n.$$

Here H_n is the so-called *Harmonic Number*, which is roughly $H_n = ln(n) + O(1)$. Thus, the two students roughly consume a logarithmic number of ice creams (in the total number of shops).

3 Soccer Betting

a) The following Markov chain models the different transition probabilities (W:Win, T:Tie, L:Loss):



b) The transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

Since the FCB has lost its previous game, the Markov chain is currently in the state L and hence, the initial vector is $q_0 = (0, 0, 1)$. The probability distribution q_3 for the game against the FC Zurich is therefore given by

$$q_3 = q_0 \cdot P^3 = (0, 0, 1) \cdot \begin{pmatrix} 0.368 & 0.248 & 0.384 \\ 0.321 & 0.256 & 0.423 \\ 0.243 & 0.248 & 0.509 \end{pmatrix} = (0.243, 0.248, 0.509).$$

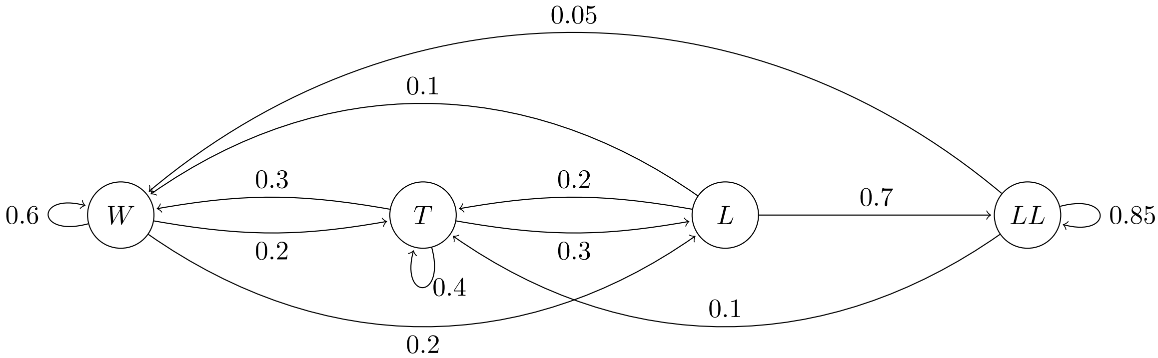
(Note that q_0 must be a row vector, not a column vector.)

Given the quotas of the exercise, the expected gains for each of the three possibilities (W, T, L) is

$$\begin{aligned} E[W] &= 0.243 \cdot 3.5 = 0.8505 \\ E[T] &= 0.248 \cdot 3.5 = 0.868 \\ E[L] &= 0.509 \cdot 2 = 1.018. \end{aligned}$$

Therefore, the best choice is to bet on a loss.

c) The new Markov chain model looks like this. In addition to the three states W, T, and L, there is now a new state LL which is reached if the team has lost twice in a row.



The new transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \tag{1}$$

As the FCB has (in this order) won and lost its last two games, the Markov chain is currently in the state $q_0 = (0, 0, 1, 0)$. The probabilities for the game against the FC Zurich can again be computed as

$$\begin{aligned} q_3 &= q_0 \cdot P^3 = (0, 0, 1, 0) \cdot \begin{pmatrix} 0.361 & 0.234 & 0.16 & 0.245 \\ 0.3105 & 0.235 & 0.15 & 0.3045 \\ 0.18175 & 0.1745 & 0.082 & 0.56175 \\ 0.42125 & 0.15475 & 0.061 & 0.6421 \end{pmatrix} \\ &= (0.18175, 0.1745, 0.082, 0.5617). \end{aligned}$$

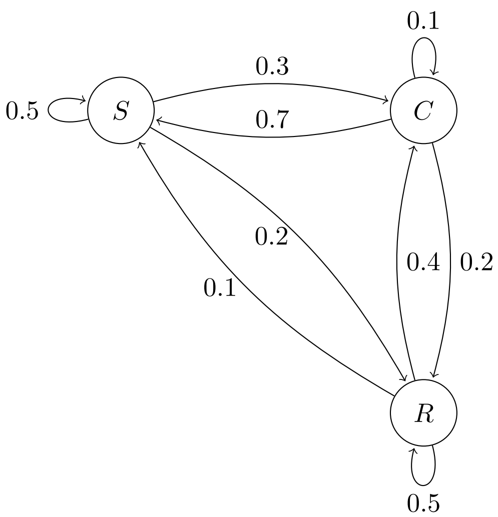
Finally, we can compute the expected win for each of the three possible bets:

$$\begin{aligned} E[W] &= 0.18175 \cdot 3.5 = 0.636125 \\ E[T] &= 0.1745 \cdot 3.5 = 0.61 \\ E[L] &= (0.082 + 0.5617) \cdot 2 = 1.2874. \end{aligned}$$

Clearly, the addition of the state LL worsens the situation for FCB.

4 The Winter Coat Problem

a) The following Markov chain models the weather situation of Robinson's island.



b) We need to determine the expected hitting time h_{SS}. Using the formula of slide 35, we obtain the following equation system:

$$
\begin{aligned}
h_{SS} &= 1 + 0.3h_{CS} + 0.2h_{RS} && (2)\\
h_{CS} &= 1 + 0.1h_{CS} + 0.2h_{RS} && (3)\\
h_{RS} &= 1 + 0.4h_{CS} + 0.5h_{RS} && (4)
\end{aligned}
$$

(2) and (3) yield that $h_{CS} = \frac{5}{6}h_{SS}$, from (2) and (4) we obtain that $h_{RS} = \frac{40}{23}h_{SS} - \frac{10}{23}$. Setting these results into (2), we obtain

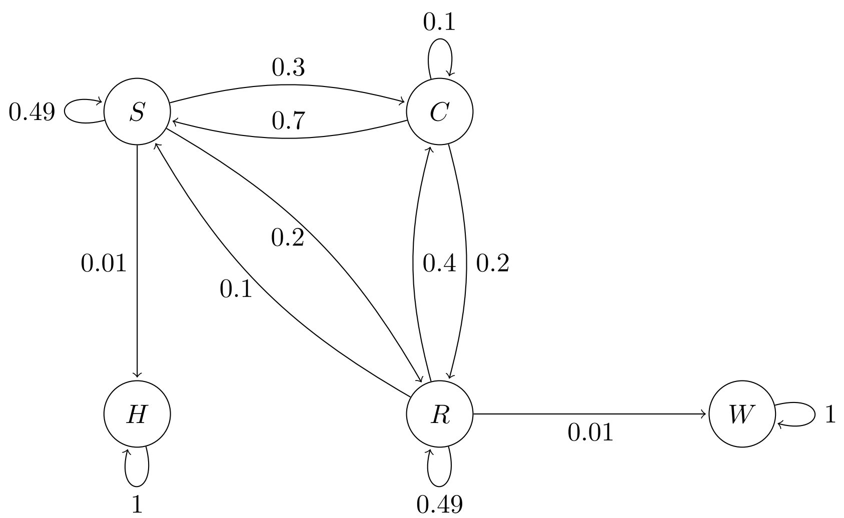
$$h_{SS} = 1 + 0.3\left(\frac{5}{6}h_{SS}\right) + 0.2\left(\frac{40}{23}h_{SS} - \frac{10}{23}\right)$$

Solve for h_{SS} to obtain

$$h_{SS} = \frac{1 - \frac{2}{23}}{1 - \frac{1}{4} - \frac{8}{23}} = \frac{84}{37} \approx 2.27$$

Thus, Mr. Robinson has to wait 2.27 days (in expectation) until having again a sunny day.

c) The modified Markov chain looks as following:



d) We need to determine the arrival probability f_{SW}, the probability that the weather will turn to winter. Using the formula of slide 35, we obtain the following equation system:

$$
\begin{aligned}
f_{SW} &= 0 + 0.3f_{CW} + 0.2f_{RW} + 0.49f_{SW} + 0.01f_{HW} && (5)\\
f_{CW} &= 0 + 0.7f_{SW} + 0.2f_{RW} + 0.1f_{CW} && (6)\\
f_{RW} &= 0.01 + 0.4f_{CW} + 0.1f_{SW} + 0.49f_{RW} && (7)\\
f_{HW} &= 0 && (8)
\end{aligned}
$$

Solving the equation system yields

$$f_{SW} = \frac{240}{619},\ f_{RW} = \frac{249}{619},\ f_{CW} = \frac{242}{619}$$

And therefore, the probability that the weather turns to winter (snowing) and Mr. Robinson needs a winter coat is $\frac{240}{619} \approx 0.39$. Note that $f_{SH} = 1 - f_{SW} = \frac{379}{619}$.