Exercise 5

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1 Regularized Luby's MIS Algorithm

Consider a regularized variant of Luby's MIS algorithm, as follows: The algorithm consists of $\log \Delta + 1$ phases, each made of $O(\log n)$ consecutive rounds. Here Δ denotes the maximum degree in the graph. In each round of the i^{th} phase, each remaining node is marked with probability $\frac{2^i}{10\Delta}$. Different nodes are marked independently. Then marked nodes who do not have any marked neighbor are added to the MIS set, and removed from the graph along with their neighbors. If at any time, a node v becomes isolated and none of its neighbors remain, then v is also added to the MIS and is removed from the graph.

(a) Argue that the set of vertices added to the MIS is always an *independent set*.

Suppose for the sake of contradiction that two neighboring nodes v and u are added to the MIS. Suppose without loss of generality that v was added no later than u, and let t be the round in which v was added to the MIS. Node u could not have been added in the same round t, because then v and u would be two neighboring marked nodes, and thus neither would be added. On the other hand, node u could not have been added in any round strictly after t, because at the end of round t, node v gets removed from the graph along with all of its neighbors, including u. Clearly, the condition that isolated nodes are added to the MIS cannot change the fact that the MIS is indeed an independent set.

(b) Prove that with high probability, by the end of the i^{th} phase, in the remaining graph each node has degree at most $\frac{\Delta}{2^i}$.

Proof is by induction. The base case $i = 0$ is trivial. Consider the time at the beginning of the i^{th} phase, and suppose that each remaining node has degree at most $\frac{\Delta}{2^{i-1}}$. Consider an arbitrary node v and suppose that v has at least $\frac{\Delta}{2^i}$ remaining neighbors, at the beginning of this phase. Per round of this phase, either at most $\frac{\Delta}{2^i} - 1$ neighbors of v remain (or v gets removed), in which case we're done, or at least $\frac{\Delta}{2^i}$ neighbors remain. In the latter case, there is a constant probability that, in the following round, a neighbor u of v 1) gets marked and 2) no neighbor of u or v except u gets marked. To be more precise, the probability that a specific neighbor u of v has these two properties is at least $\frac{2^i}{10\Delta} \cdot (1 - \frac{2^i}{10\Delta})^{2 \cdot \frac{\Delta}{2^{i-1}}} \geq \frac{2^i}{10\Delta} \cdot 4^{-2/5} \geq \frac{1}{2} \cdot \frac{2^i}{10\Delta}$, which implies that the probability that at least one neighbor of v has these properties is at least $\frac{\Delta}{2^i} \cdot \frac{1}{2} \cdot \frac{2^i}{10\Delta} = \frac{1}{20}$. (Note that it cannot be the case that two different neighbors of v both satisfy these properties; hence, we can simply add up the probabilities for the different neighbors as we did.) We conclude that the probability that v is removed in the considered round (due to having a neighbor selected into the MIS) is at least $1/20$; more precisely, in each round of the i^{th} phase in which node v has degree at least $\frac{\Delta}{2^i}$, node v gets removed with probability at least $1/20$.

Thus, choosing the constant hidden in the O-notation to be 100, and considering the $100 \log n$ rounds of the phase, the probability of v remaining with degree above $\frac{\Delta}{2^i}$ at the end of the i^{th} phase is at most $(1 - 1/20)^{100 \log n} \leq 2^{-5 \log n} = 1/n^5$. A union bound over all such nodes v shows that with probability $1 - 1/n^4$, no such node with degree above $\frac{\Delta}{2^i}$ remains by the end of the i^{th} phase (assuming that each remaining node at the beginning of the i^{th} phase has degree at most $\frac{\Delta}{2^{i-1}}$). Since we have $\log \Delta + 1$ phases, and in each phase the probability that something goes wrong (and that in all previous phases nothing went wrong) is upper bounded by $1/n^4$, the probability that something goes wrong in at least one phase is at most $\frac{\log \Delta + 1}{n^4} \leq \frac{1}{n^3}$. Hence, the statement we want to prove indeed holds with high probability.

(c) Conclude that the set of vertices added to the MIS is a *maximal* independent set, with high probability.

By what we proved in part (b), by the end of phase $\log \Delta + 1$, each node's degree must be at most $\frac{\Delta}{2\Delta} = 1/2$, with high probability. This means the degree is actually 0. Once a node reaches degree 0, it gets added to the MIS. If the node v was removed anytime before that, it must have been that there was a node u in the neighborhood of v (i.e., either $u = v$ or u is a neighbor of v) that was added to the MIS. Hence, the set is also maximal, with high probability.

(d*) Reprove item (b) assuming only pairwise independence between the marking events of different nodes, in the same round.

2 Randomized Coloring Algorithm

Consider the following simple randomized $\Delta + 1$ coloring algorithm: Per round, each node selects one of the colors not already *taken away* by its neighbors, at random. Then, if v selected a color and none of its neighbors selected the same color in that round, v gets colored with this color and takes this color away permanently. That is, none of the neighbors of v will select this color in any of the future rounds.

(a) Prove that in the first round, each node has at least a constant probability of being colored.

Consider a node v and let c be the color that v selects in the first round. The probability that none of the neighbors of v selects c is at least $(1 - \frac{1}{\Delta+1})^{\Delta} \geq 1/4$.

(b*) Prove that per round, each remaining node has at least a constant probability of being colored.

(b') If item (b) turns out to be complex, you may assume that we use $\lceil 1.02\Delta \rceil$ colors, instead of $\Delta + 1$. Prove that per round, each remaining node has at least a constant probability of being colored.

Consider an arbitrary round v and suppose that by the end of the previous round, exactly d neighbors of v remain uncolored. Then, the number of colors remaining unblocked for v is at least $1.02\Delta - (\Delta - d) \geq d + 0.02\Delta \geq 1.02d$. Let us say a remaining color c is bad in this round if at least one of the remaining neighbors of v selects c in this round. Regardless of how the remaining neighbors of v select their colors in this round, at most d colors are bad. Hence, at least a $\frac{1.02d-d}{1.02d} > 0.01$ fraction of colors are good. If v picks a good color, it becomes colored permanently. Hence, v has a probability of at least 0.01 of picking a good color and getting permanently colored in this round.

(c) Conclude that within $O(\log n)$ rounds, all nodes are colored, with high probability.

In each round, each remaining node v gets colored with probability at least 0.01. Thus, the probability that we go for $200 \log n$ rounds and still v remains uncolored is at most $(1-1/100)^{200 \log n} \leq 2^{-2 \log n} = 1/n^2$. By a union bound over all vertices v, we get that with probability at least $1-1/n$, no node remains uncolored by the end of round $200 \log n$.