# Principles of Distributed Computing Exercise 7: Sample Solution 

## 1 Concurrent Ivy

a) The three nodes are served in the order $v_{2}, v_{3}, v_{1}$.
b) Figure 1 depicts the structure of the tree after the requests have been served. Since $v_{1}$ is served last, it is the holder of the token at the end.


Figure 1: Tree after the requests have been served.

## 2 Tight Ivy

In order to show that the bound of $\log n$ steps on average is tight, we construct a special tree which is defined recursively as follows. The tree $\mathcal{T}_{0}$ consists of a single node. The tree $\mathcal{T}_{i}$ consists of a root together with $i$ subtrees, which are $\mathcal{T}_{0}, \ldots, \mathcal{T}_{i-1}$, rooted at the $i$ children of the root, see Figure 2.

First, we will show that the number of nodes in the tree $\mathcal{T}_{i}$ is $2^{i}$. This obviously holds for $\mathcal{T}_{0}$. The induction hypothesis is that it holds for all $\mathcal{T}_{0}, \ldots, \mathcal{T}_{i-1}$. It follows that the number of nodes of $\mathcal{T}_{i}$ is $n=1+\sum_{j=0}^{i-1} 2^{j}=2^{i}$.

We will show now that the radius of the root of $\mathcal{T}_{i}$ is $\mathcal{R}\left(\mathcal{T}_{i}\right)=i$. Again, this is trivially true for $\mathcal{T}_{0}$. It is easy to see that $\mathcal{R}\left(\mathcal{T}_{i}\right)=1+\mathcal{R}\left(\mathcal{T}_{i-1}\right)$, because $\mathcal{T}_{i-1}$ is the child with the largest radius. Inductively, it follows that $\mathcal{R}\left(\mathcal{T}_{i}\right)=i$.

By definition, when cutting off the subtree $\mathcal{T}_{i-1}$ from $\mathcal{T}_{i}$, the resulting tree is again $\mathcal{T}_{i-1}$. Let $\mathcal{C}: \mathcal{T}_{i} \mapsto \mathcal{T}_{i-1}$ denote this cutting operation. For all $i>0$, we thus have that $\mathcal{C}\left(\mathcal{T}_{i}\right)=\mathcal{T}_{i-1}$. We will now start a request at the single node $v$ with a distance of $i$ from the root in $\mathcal{T}_{i}$. On its path to


Figure 2: The trees $\mathcal{T}_{0}, \ldots, \mathcal{T}_{3}$.
the root, the request passes nodes that are roots of the trees $\mathcal{T}_{1}, \ldots, \mathcal{T}_{i}$. All of those nodes become children of the new root $v$ according to the Ivy protocol. The new children lose their largest "child" subtree in the process, thus the children of node $v$ have the structures $\mathcal{C}\left(\mathcal{T}_{1}\right), \ldots, \mathcal{C}\left(\mathcal{T}_{i}\right)=$ $\mathcal{T}_{0}, \ldots, \mathcal{T}_{i-1}$. Hence, the structure of the tree does not change due to the request and all subsequent requests can also cost $i$ steps. Since $n=2^{i}$, each request costs exactly $\log n$.

