Election vs. Selection: How Much Advice is Needed to Find the Largest Node in a Graph?

Avery Miller
University of Manitoba
avery@averymiller.ca

Andrzej Pelc
Université du Québec en Outaouais
andrzej.pelc@uqo.ca

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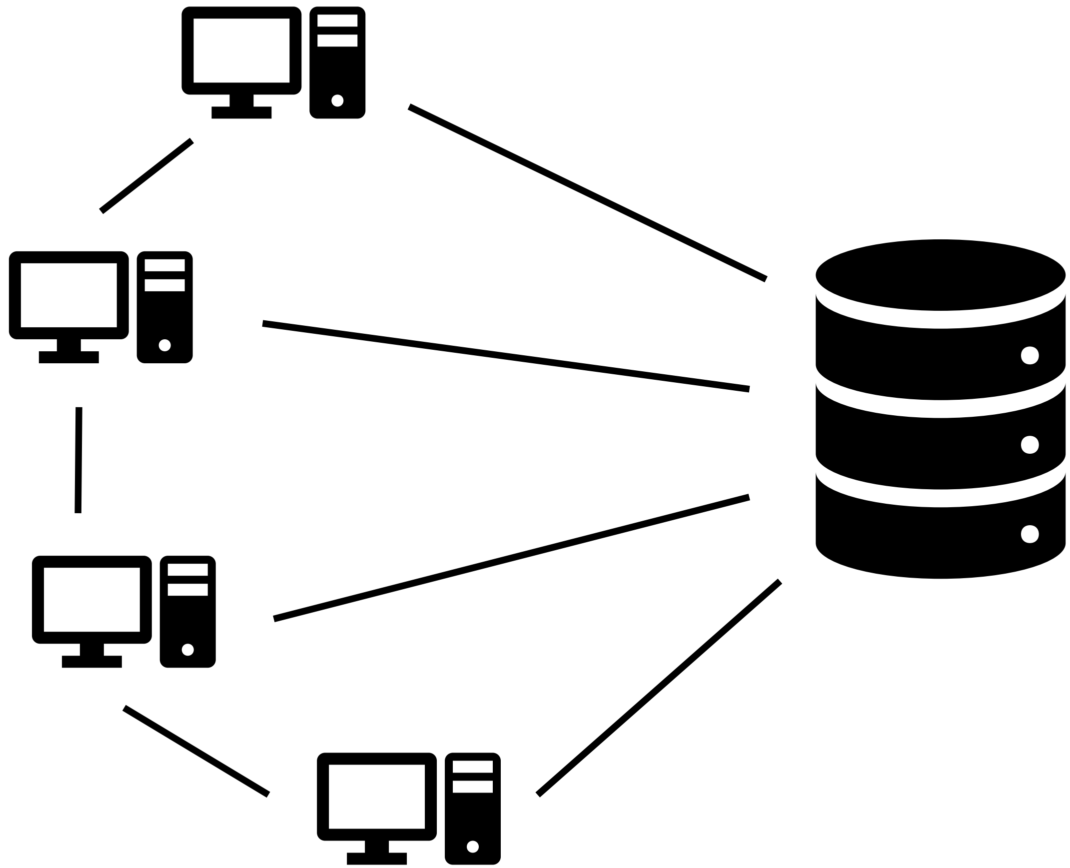
ETHZ - Seminar in Distributed Computing FS 2018

Application – Shared Resource

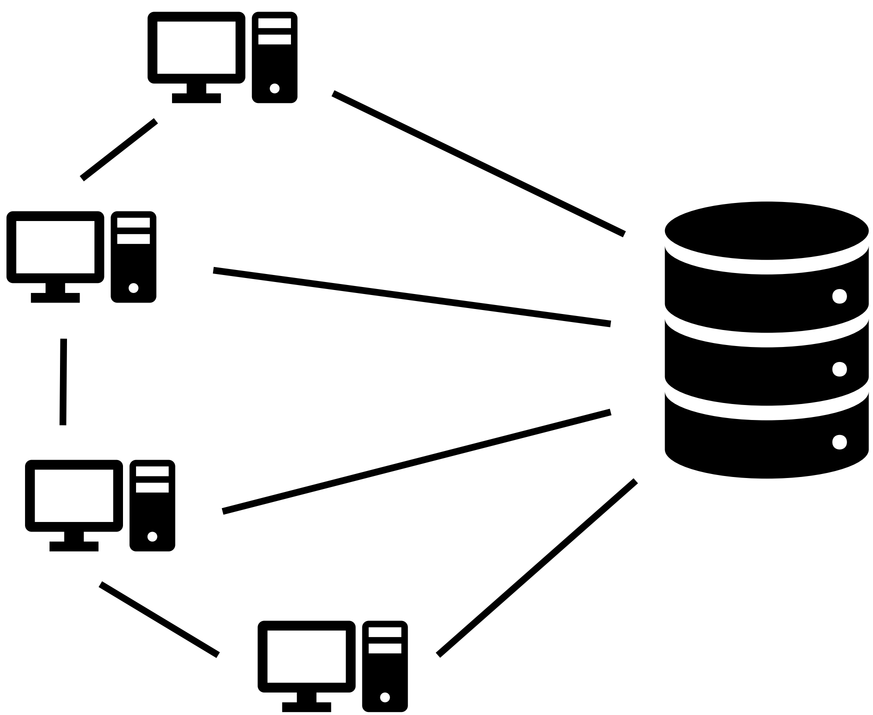


Microsoft Docs, **Leader Election Pattern,** 23.06.2017
https://docs.microsoft.com/en-us/azure/architecture/patterns/leader-election

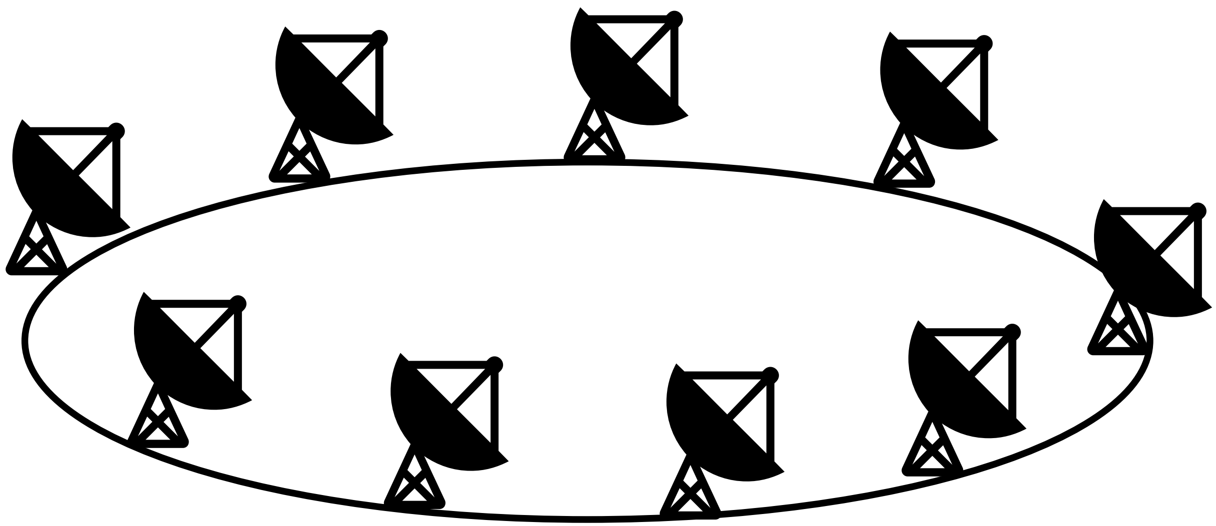
Application – Shared Resource

Access to shared resource -> need coordinator

Failure resilience -> need new leader

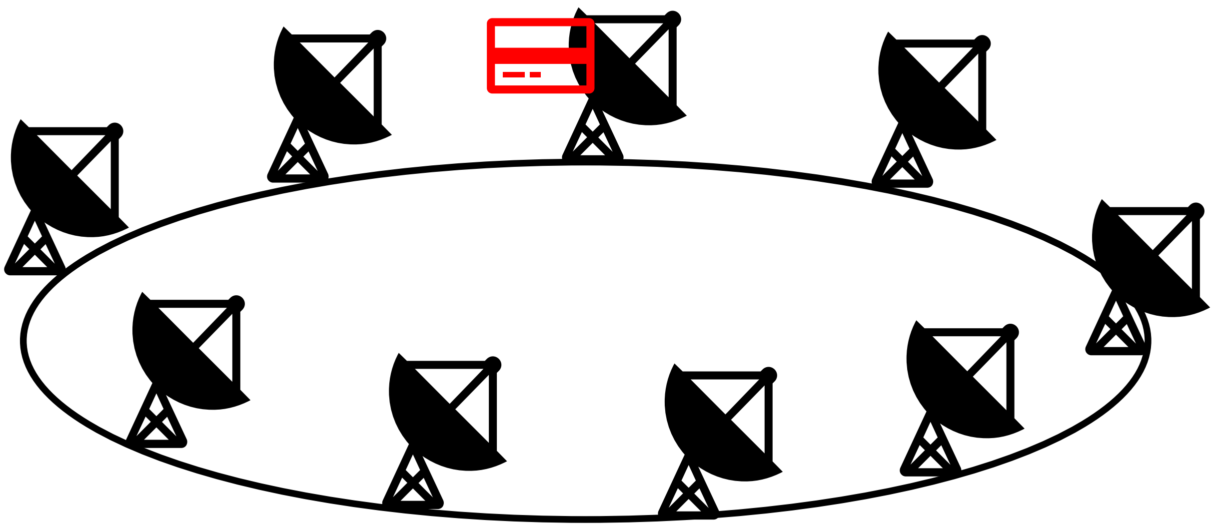


Application – Radiocom



G. Le Lann, **Distributed Systems - Towards a Formal Approach**
Proc. IFIP Congress, 1977, 155-160, North Holland.

Application – Radiocom



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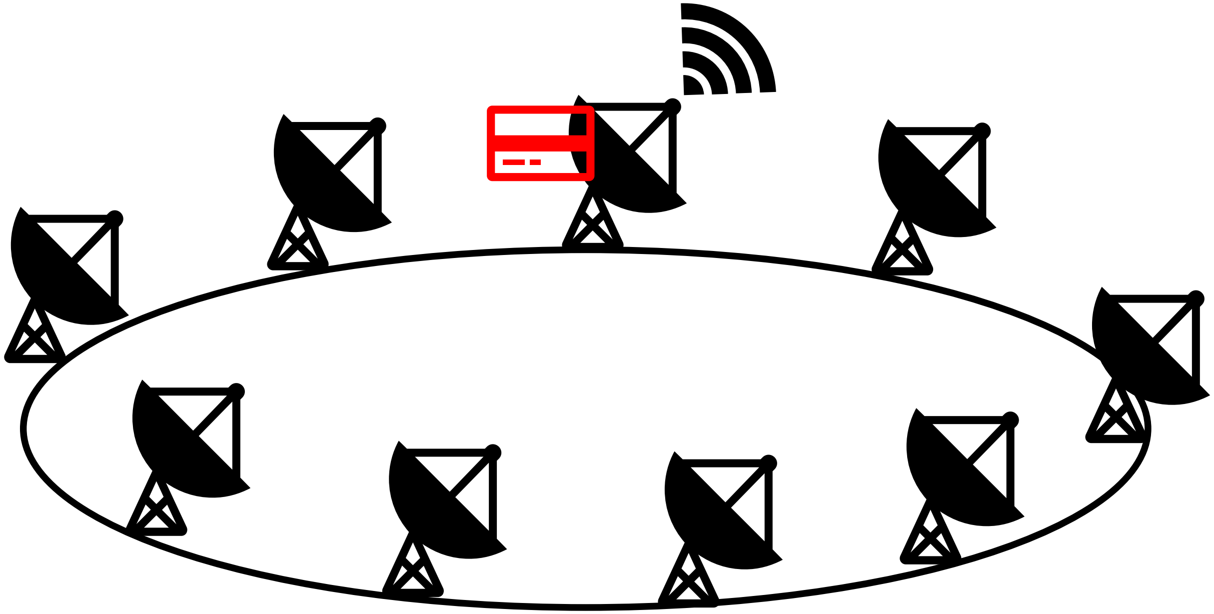

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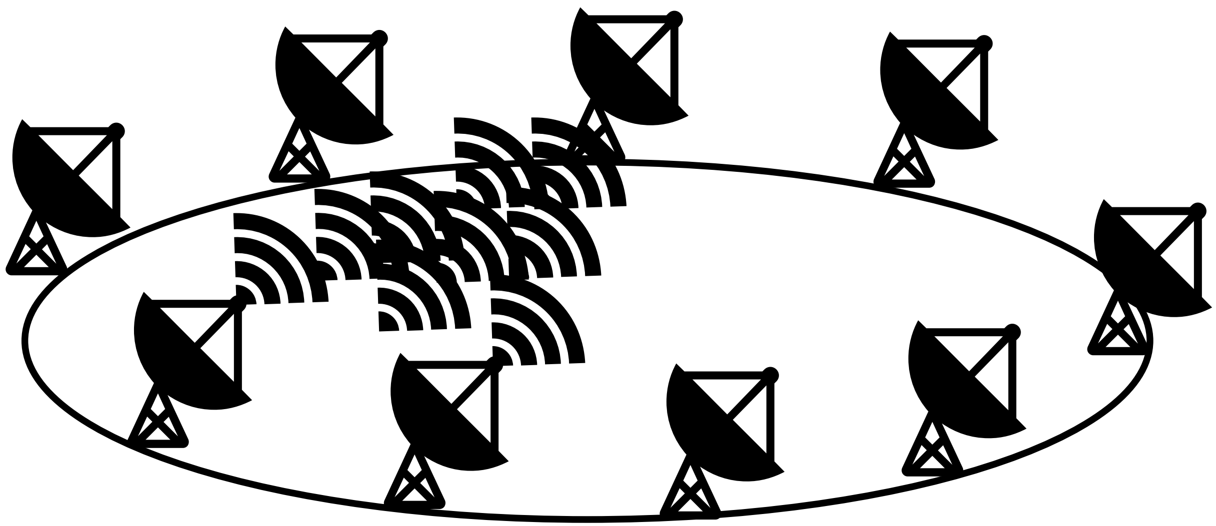

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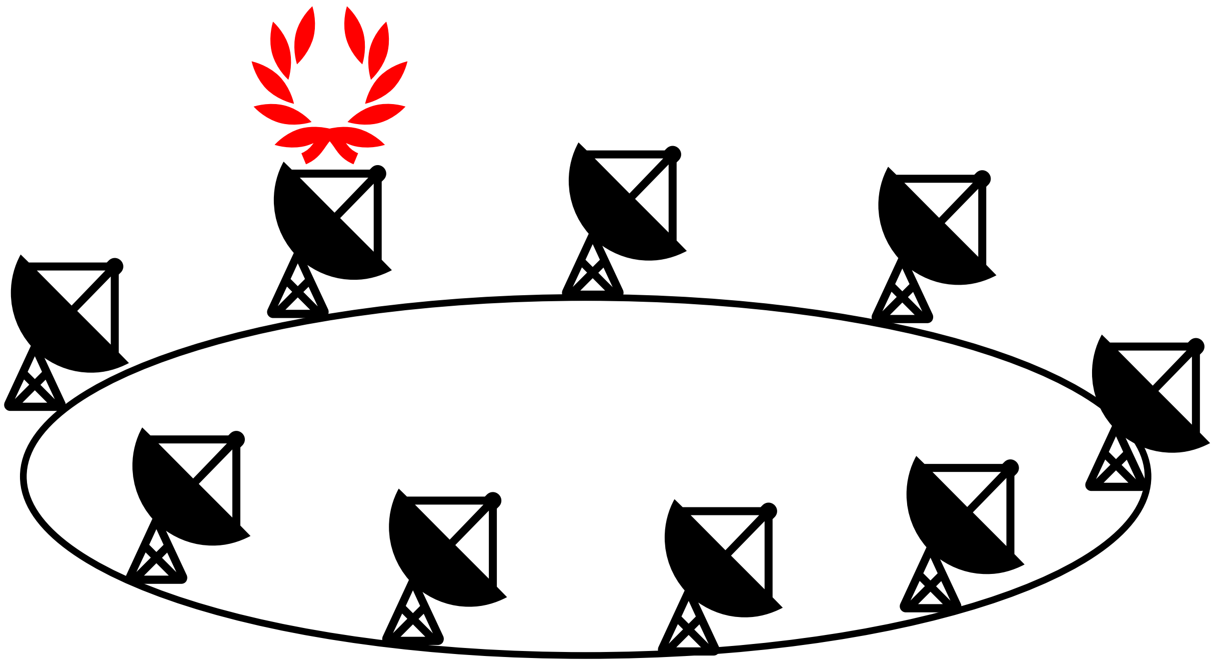

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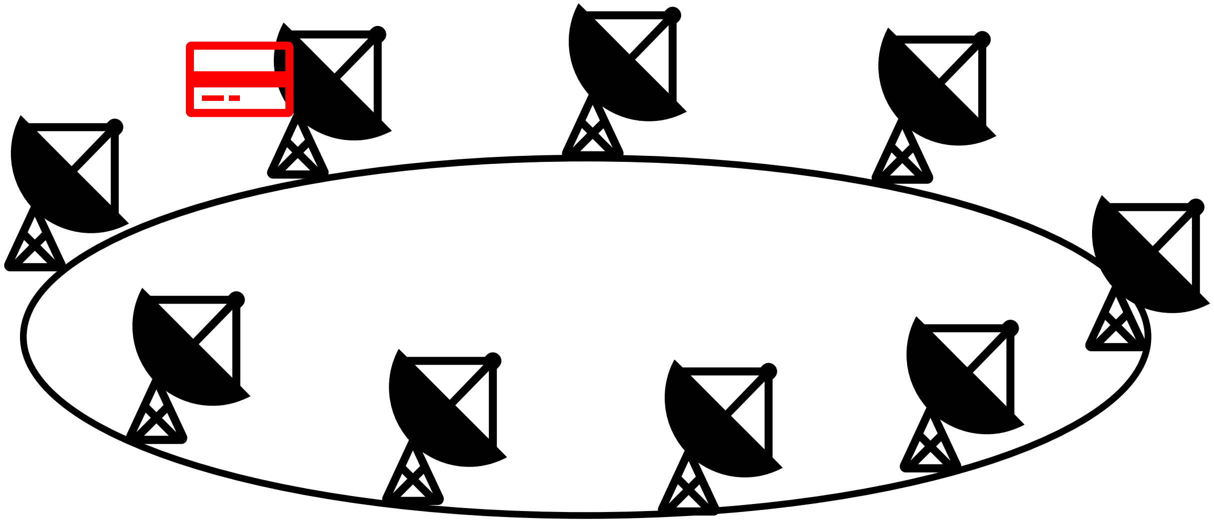
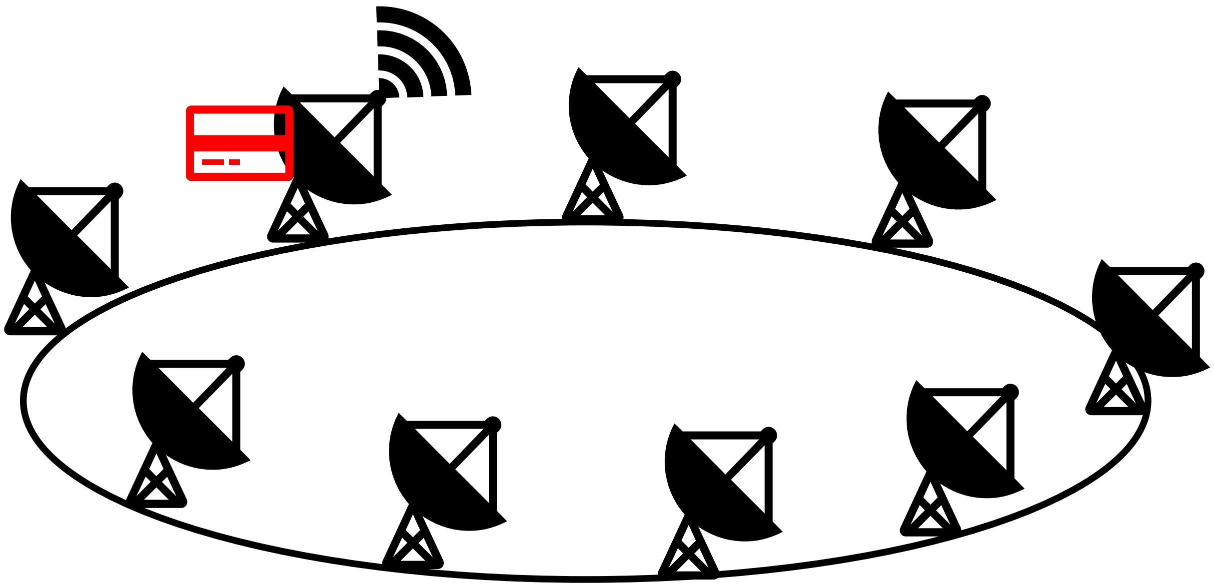


Application – Radiocom



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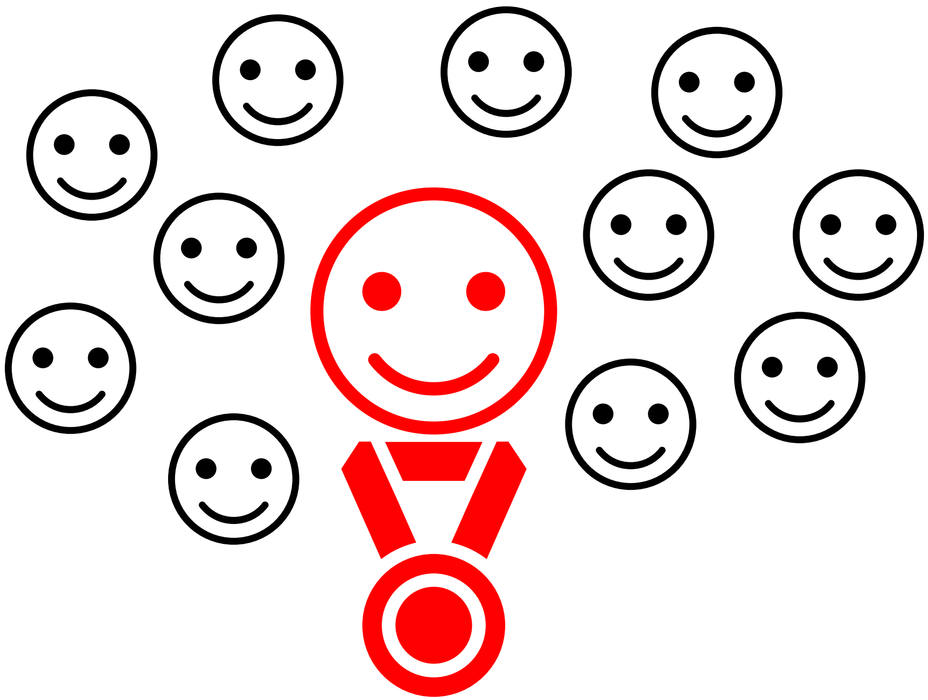
Election

Find a **leader**

Everyone knows its identity



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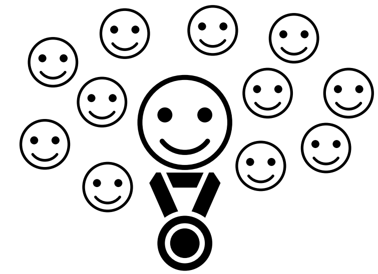

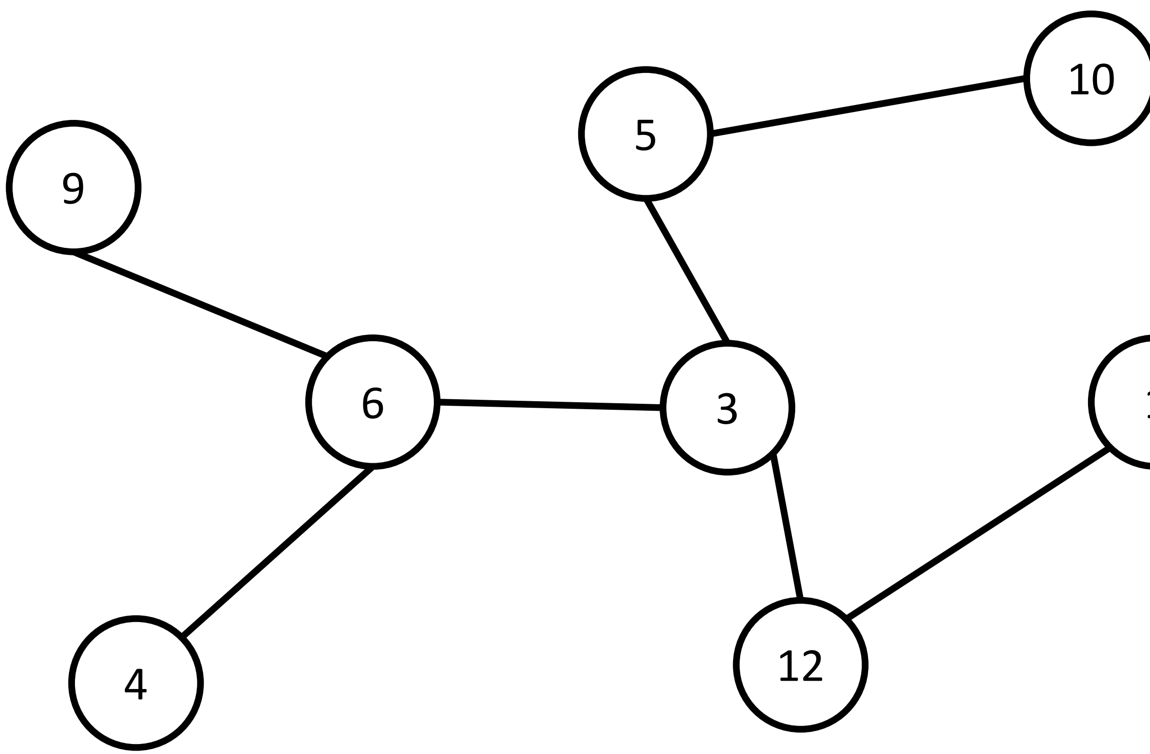

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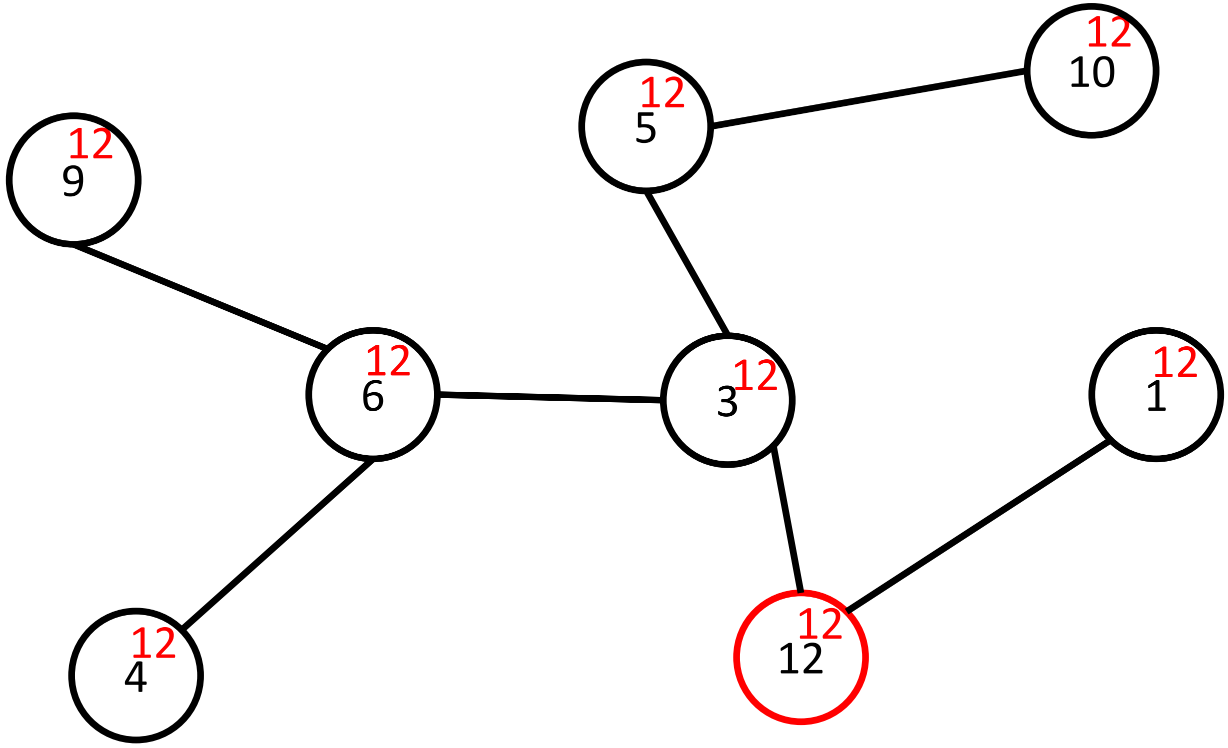
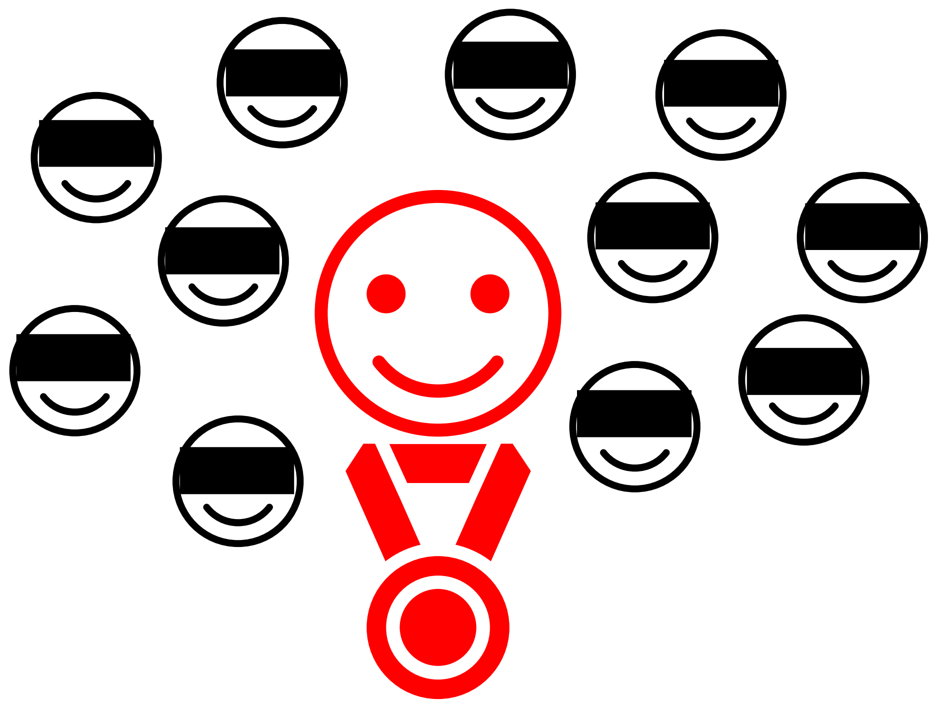


Selection

Leader outputs **1**

Every other node outputs **0**



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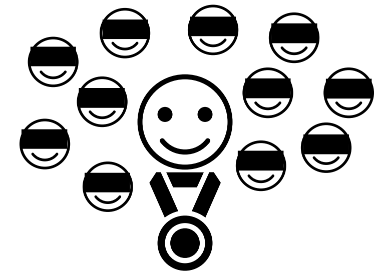

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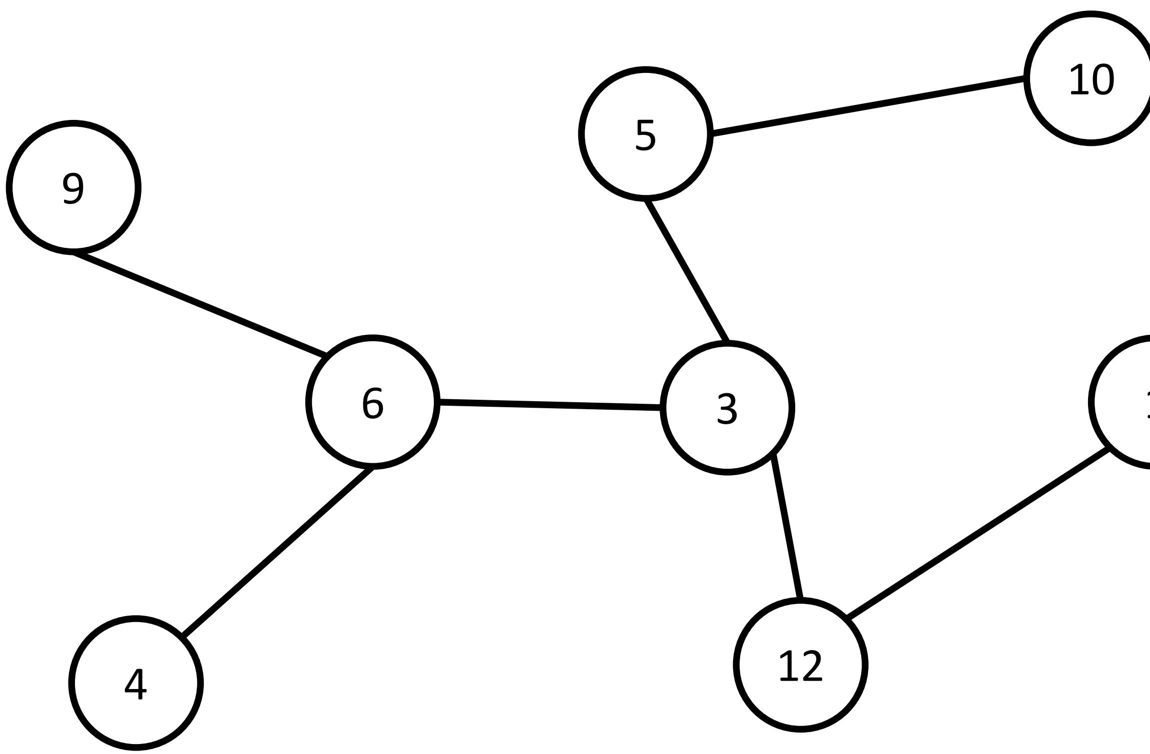

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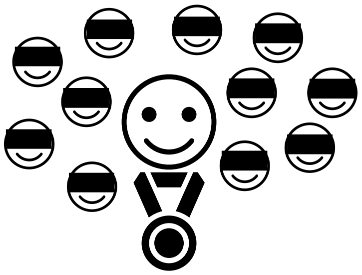

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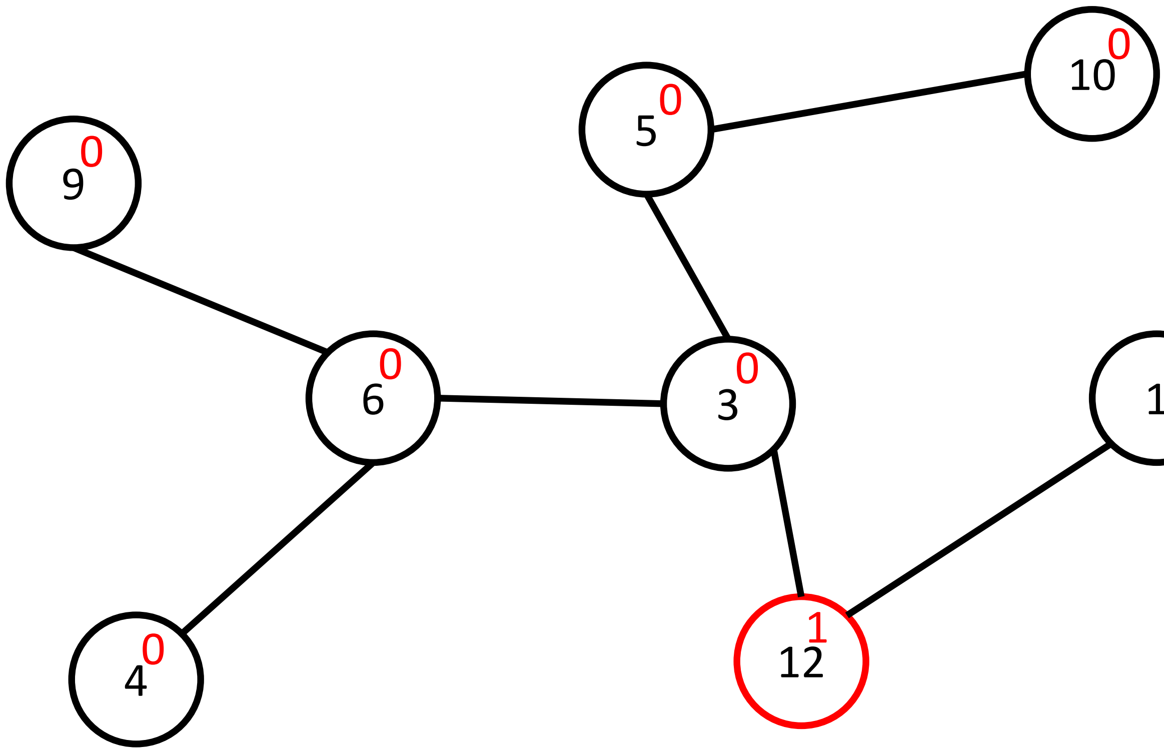
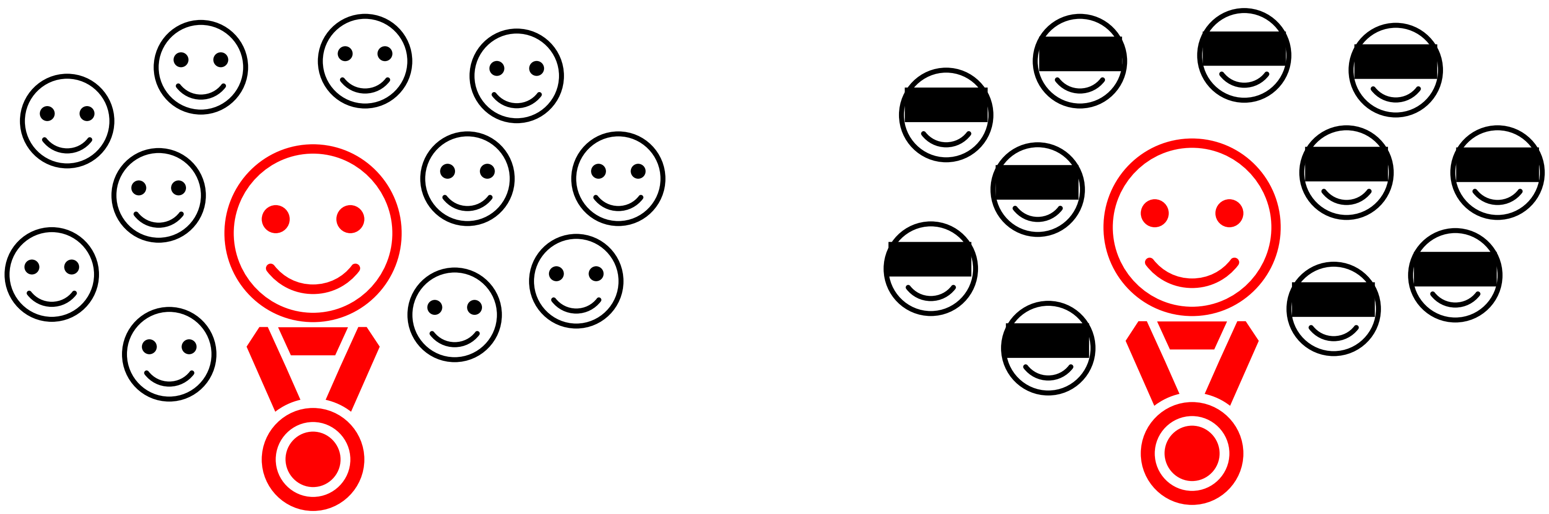


Election vs Selection

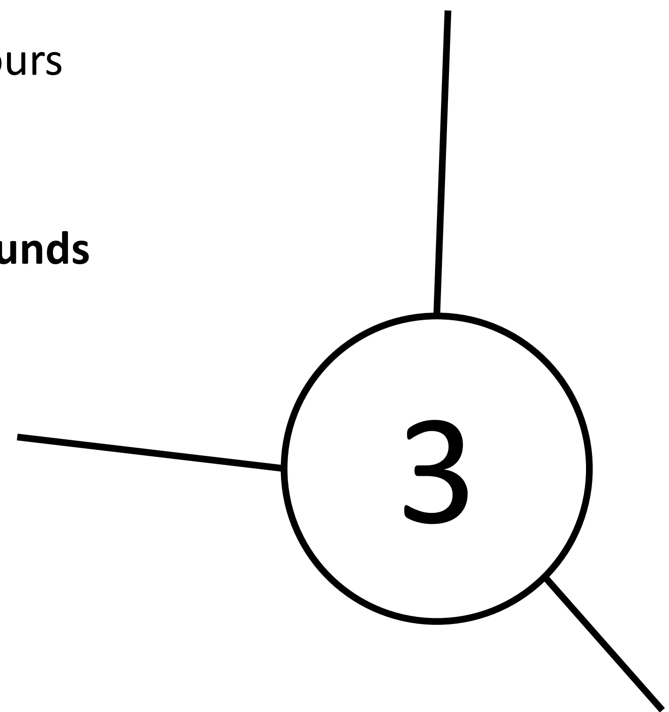


Synchronous Algorithms

In each **round**:

- **Send** messages to neighbours
- **Recei**ve messages from neighbours
- Do some **computation**

Time complexity is the **number of rounds**



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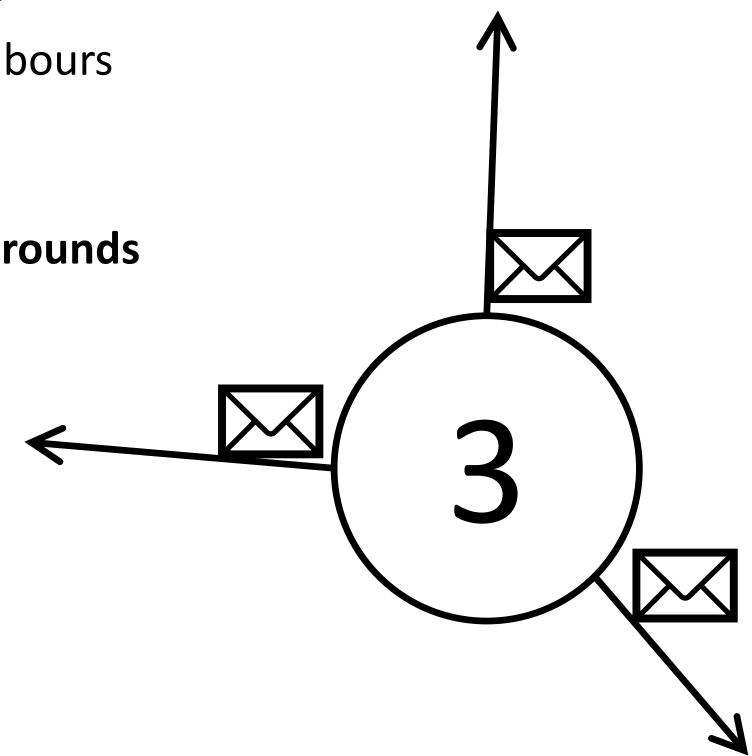

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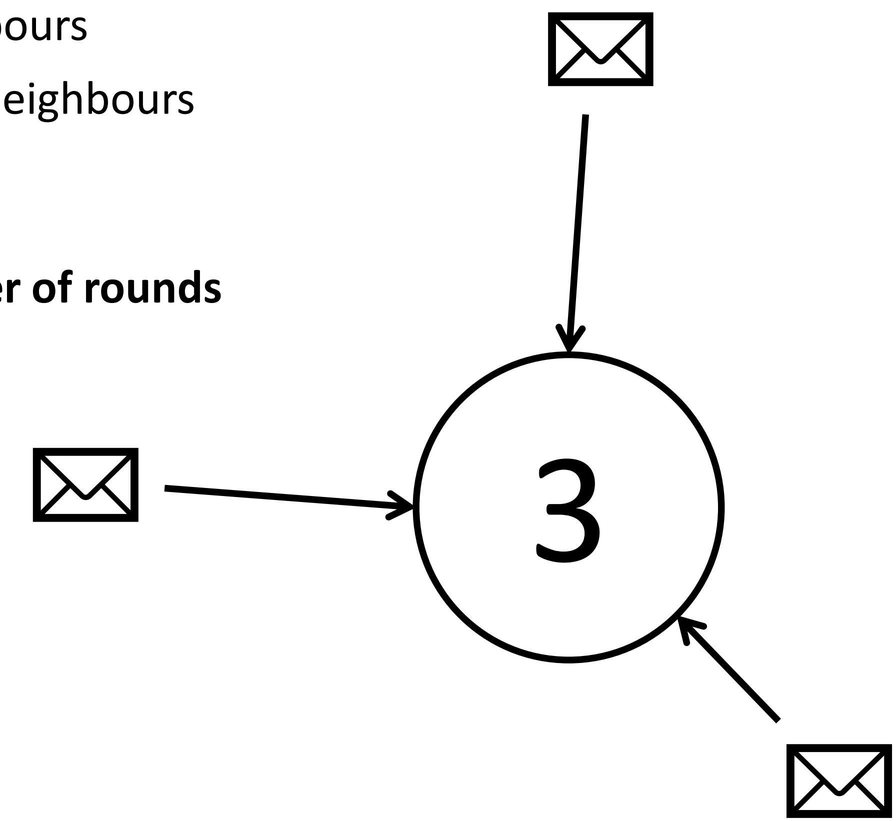

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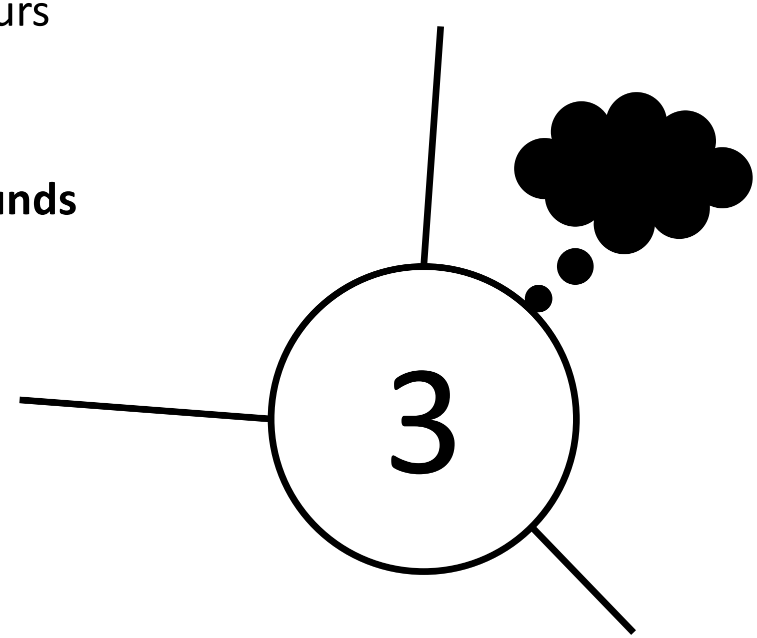
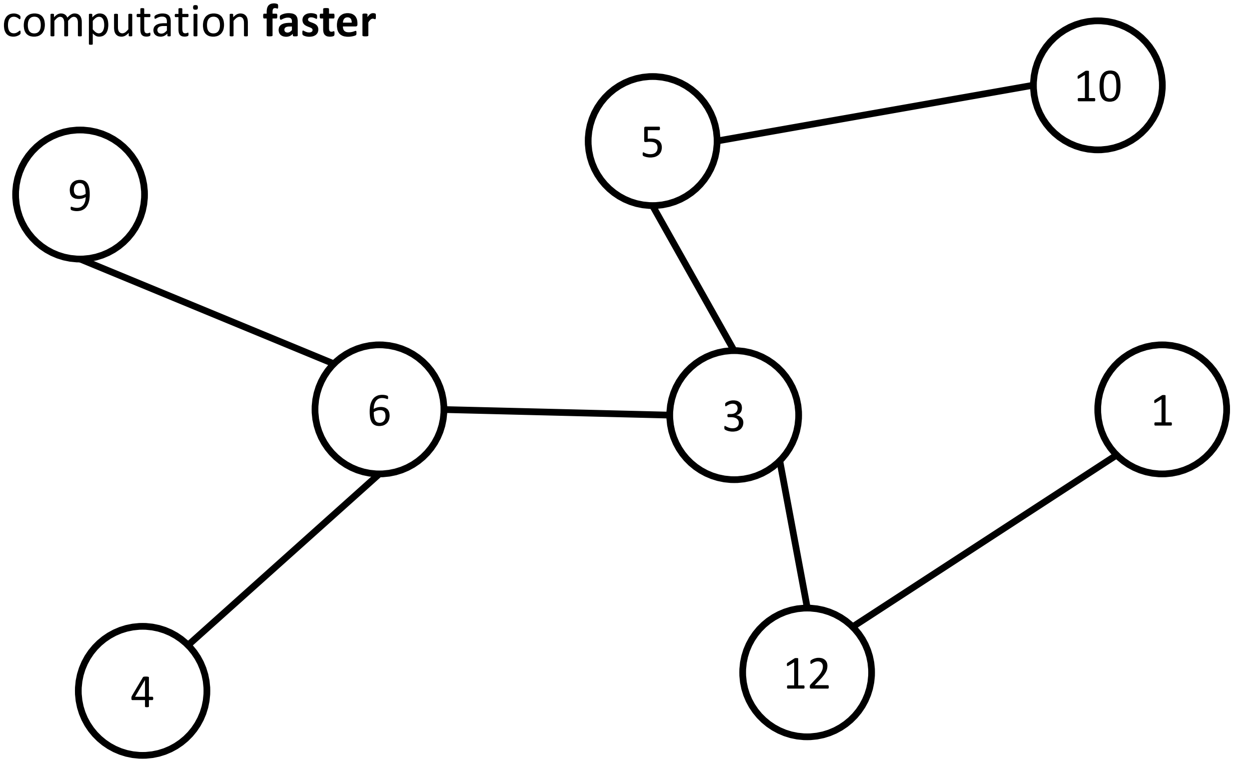


Algorithm with advice

Oracle with **full knowledge**

Gives same **advice** to each node

Goal: make computation **faster**



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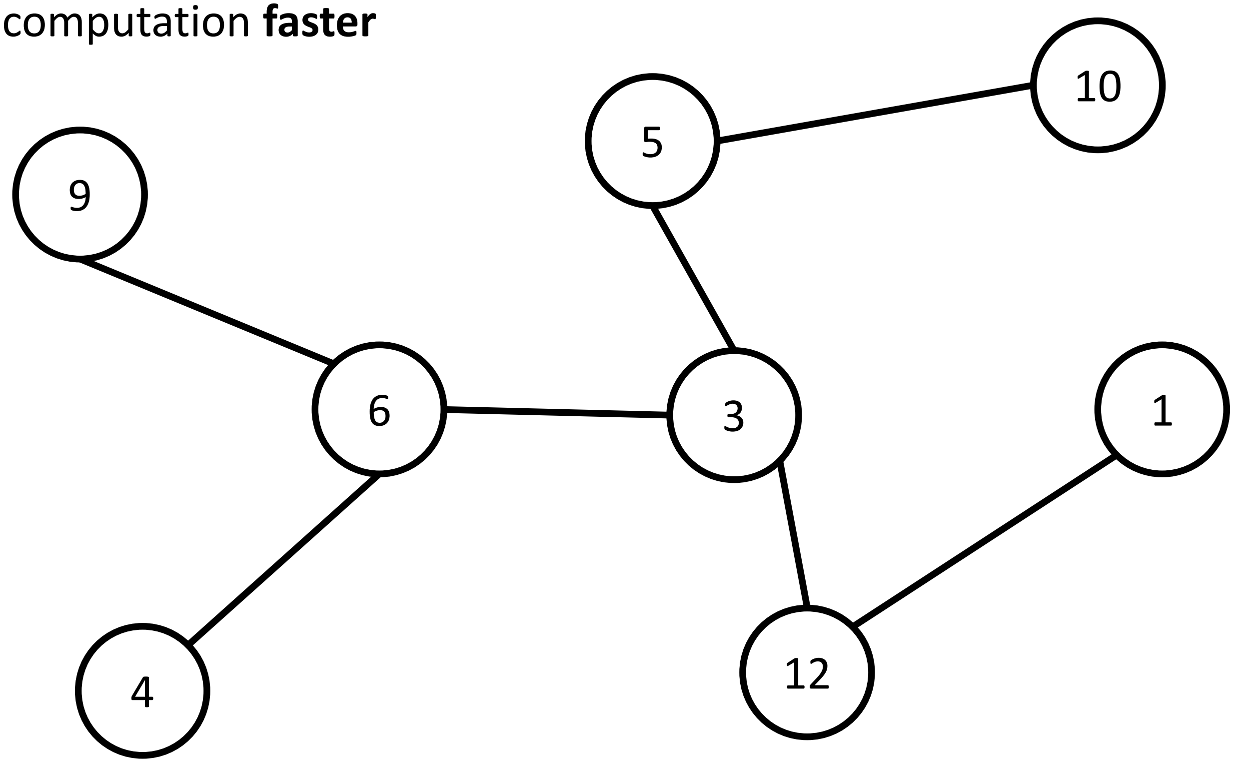

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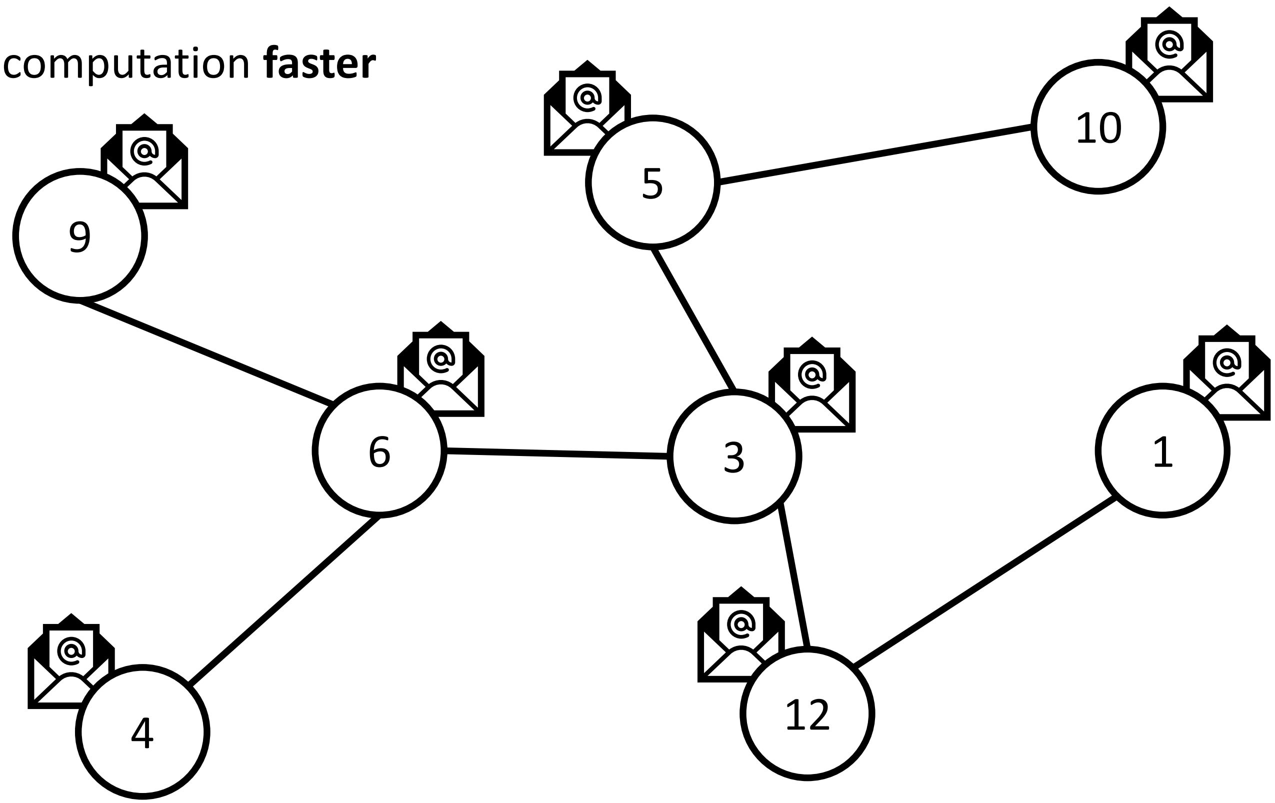
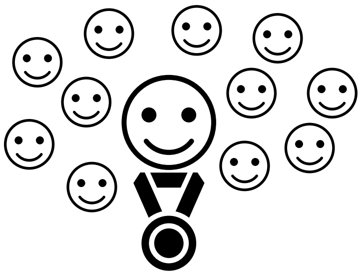


Algorithm with advice

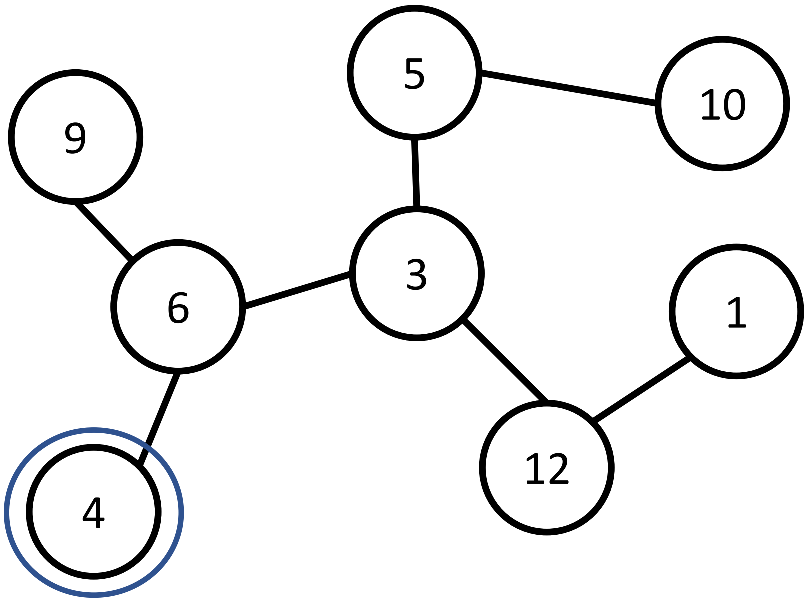


Example : Election **without** advice

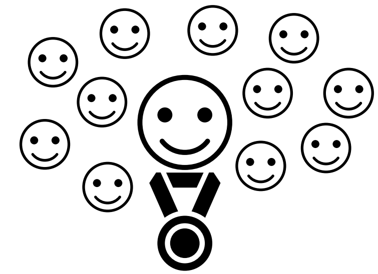
Notations:

$K(r, v) =$ knowledge of v after r rounds

$\Lambda(r, v)$ set of labels induced by $K(r, v)$



Algorithm with advice



Example : Election **without** advice

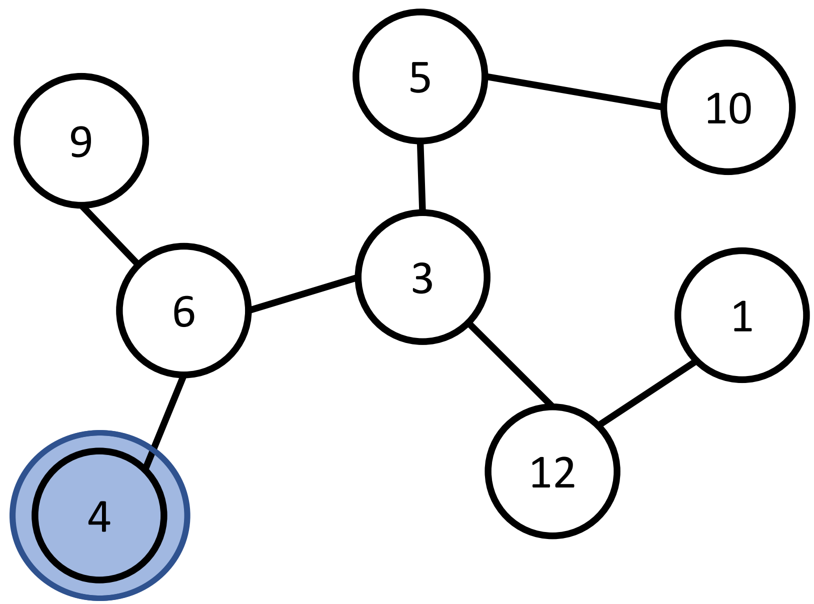
Notations:

$K(r, v)$ = knowledge of v after r rounds

$\Lambda(r, v)$ set of labels induced by $K(r, v)$

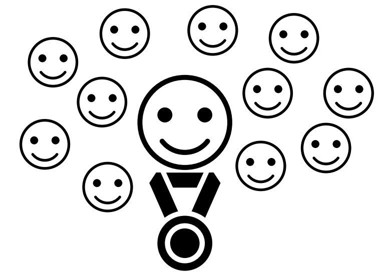
$K(0, 4)$

$\Lambda(0, 4) = \{4\}$



Round 0

Algorithm with advice



Example : Election **without** advice

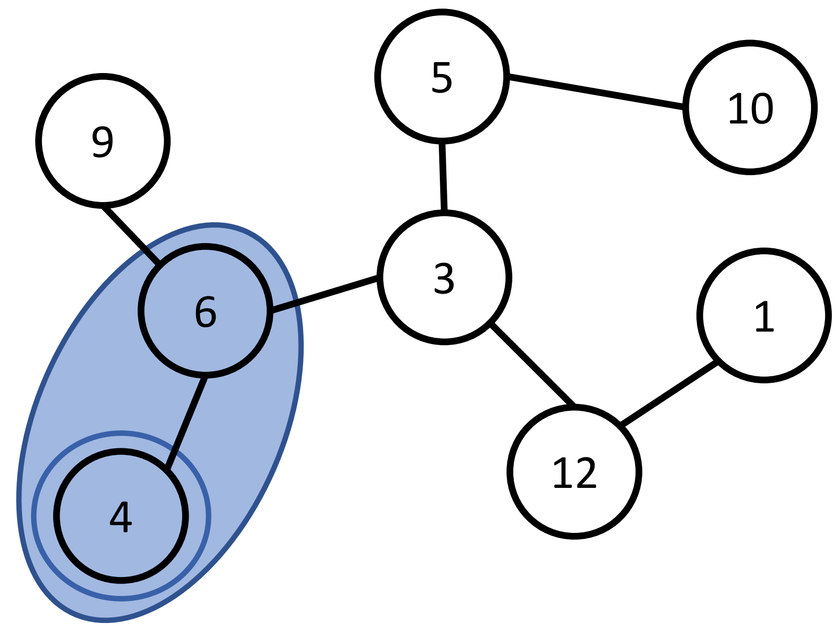
Notations:

$K(r, v)$ = knowledge of v after r rounds

$\Lambda(r, v)$ set of labels induced by $K(r, v)$

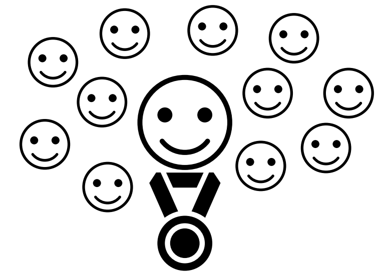
$K(1, 4)$

$\Lambda(1, 4) = \{4, 6\}$



Round 1

Algorithm with advice



Example : Election **without** advice

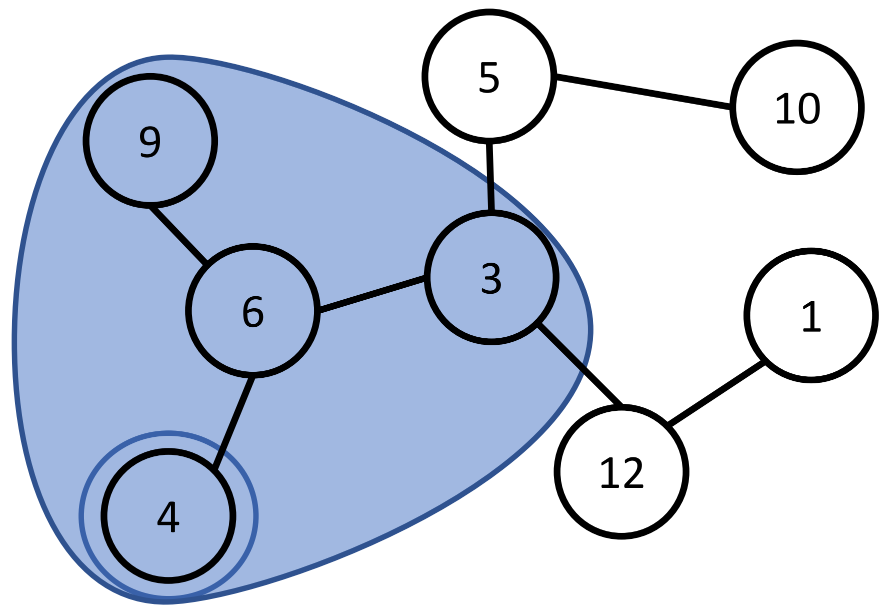
Notations:

$K(r, v)$ = knowledge of v after r rounds

$\Lambda(r, v)$ set of labels induced by $K(r, v)$

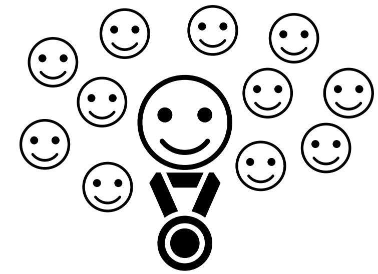
$K(2, 4)$

$\Lambda(2, 4) = \{3, 4, 6\}$



Round 2

Algorithm with advice



Example : Election **without** advice

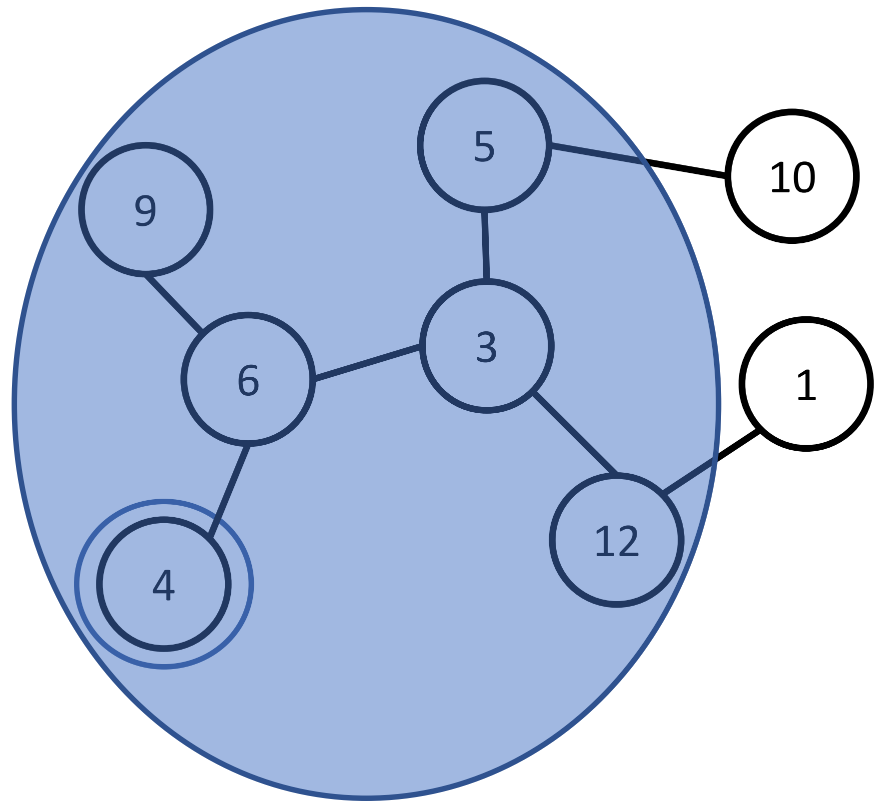
Notations:

$K(r, v)$ = knowledge of v after r rounds

$\Lambda(r, v)$ set of labels induced by $K(r, v)$

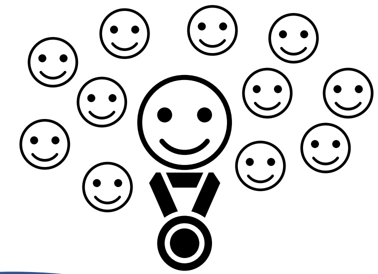
$K(3,4)$

$\Lambda(3,4) = \{3, 4, 5, 6, 12\}$



Round 3

Algorithm with advice



Example : Election **without** advice

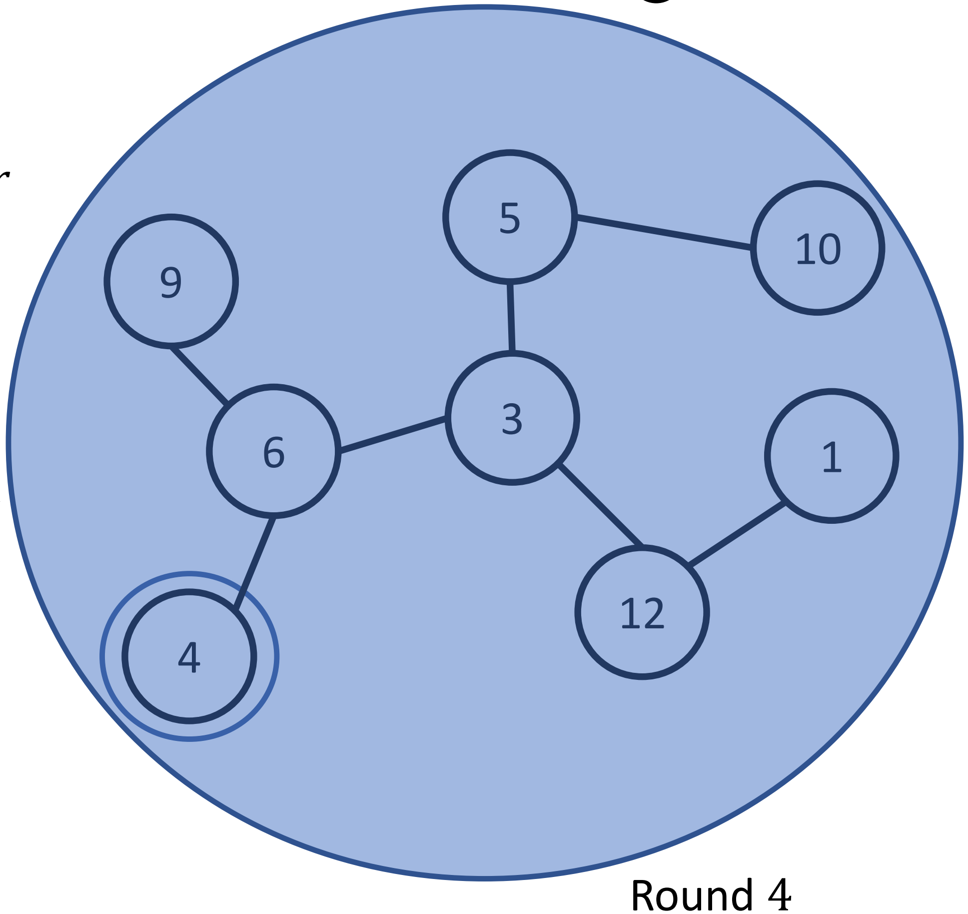
Notations:

$K(r, v)$ = knowledge of v after r rounds

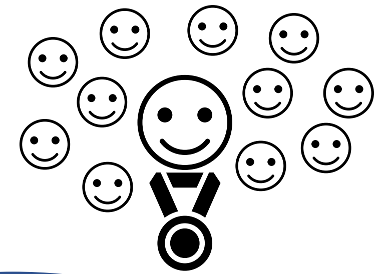
$\Lambda(r, v)$ set of labels induced by $K(r, v)$

$K(4,4)$

$\Lambda(4,4) = \{1, 3, 4, 5, 6, 10, 12\}$



Algorithm with advice

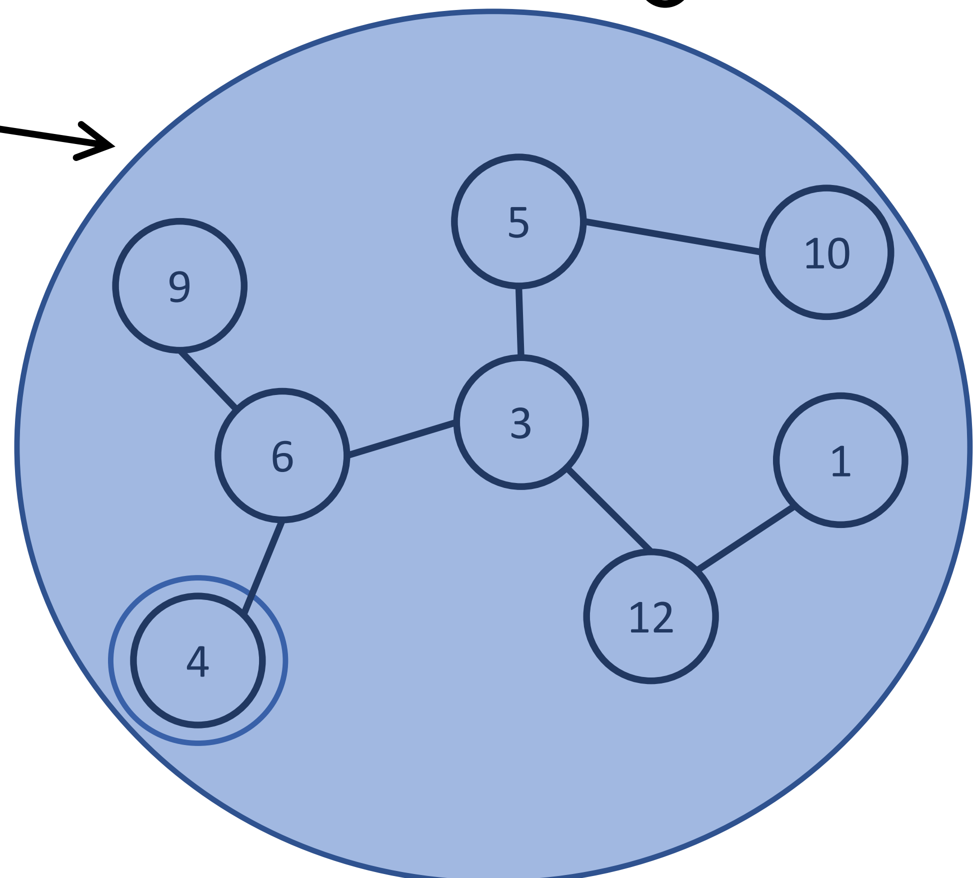


Example : Election **without** advice

$K(4,4)$

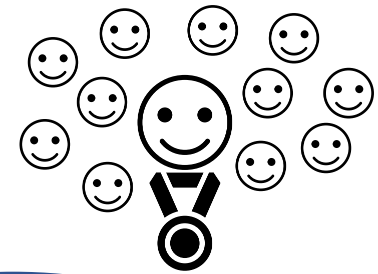
$\Lambda(4,4) = \{1,3,4,5,6,10,12\}$

Requires another round
to terminate!



Round 4

Algorithm with advice



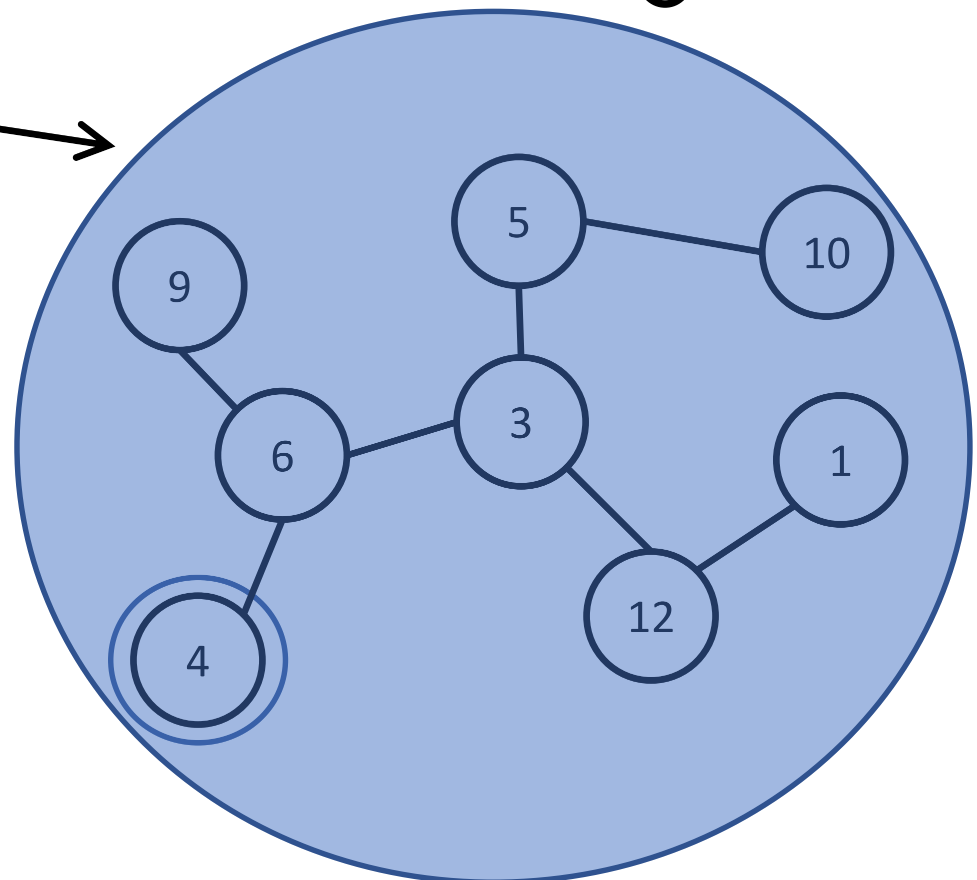
Example : Election **without** advice

$K(4,4)$

$\Lambda(4,4) = \{1,3,4,5,6,10,12\}$

Requires another round
to terminate!

What if we give some **advice**?



Round 4

Task – Measure of Difficulty

Time constraint for the execution

How much **advice** needed ?

Upper and **lower** bound the size of advice

Tight Bounds on Advice

Tight bounds are given on the size of advice.

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$$\Theta\big(f(x)\big) \;\Leftrightarrow\; \Omega\big(f(x)\big) \wedge O\big(f(x)\big)$$

Tight Bounds on Advice

Tight bounds are given on the size of advice.

$$\Theta\big(f(x)\big) \iff \Omega\big(f(x)\big) \wedge \mathrm{O}\big(f(x)\big)$$

Lower bound l

Find a class of graphs for which a least l advice needed for any algorithm

Upper bound u

Find an algorithm for which at most u advice needed on all graphs

How is it helpful?

Can **rule out** entire classes of algorithms

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Given result: Task T needs $\Theta(\log n)$ bits of advice

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Given result: Task T needs $\Theta(\log n)$ bits of advice

Proposed algorithm: Needs linear upper bound on n as advice.

How is it helpful?

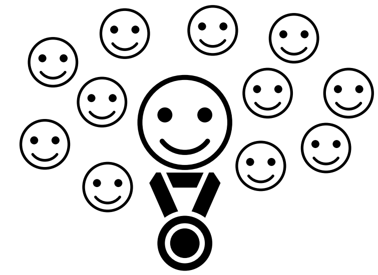
Can **rule out** entire classes of algorithms

Given result: Task T needs $\Theta(\log n)$ bits of advice

Proposed algorithm: Needs linear upper bound on n as advice.

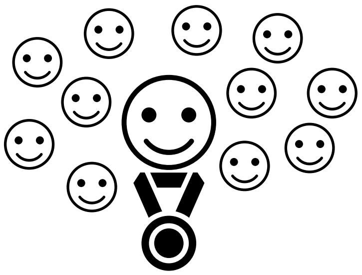
Contradiction: Advice can be given by $\lceil \log n \rceil$, using $\Theta(\log \log n)$ bits.

Results - Election



Time	Advice
$> \ diam$	0
$diam$	$\Theta(\log diam)$
$< \ diam$	$\Theta(\log n)$

Results - Election



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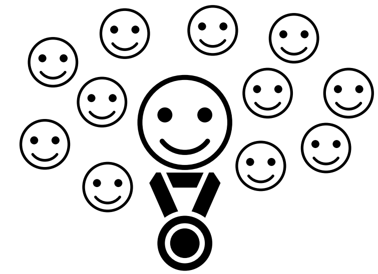
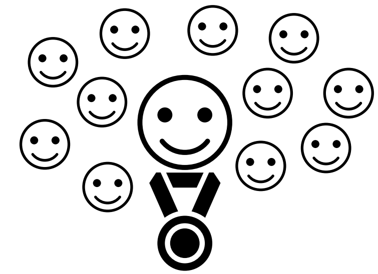


Time	Advice
$> \ diam$	0
$diam$	$\Theta(\log diam)$
$< \ diam$	$\Theta(\log n)$

Provide the **diameter** of the graph

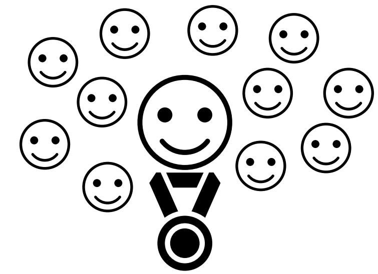
Results - Election



Time	Advice
$> \ diam$	0
$diam$	$\Theta(\log diam)$
$< \ diam$	$\Theta(\log n)$

No better advice than to **give** the **solution**

Results - Election



Time	Advice
$> \ diam$	0
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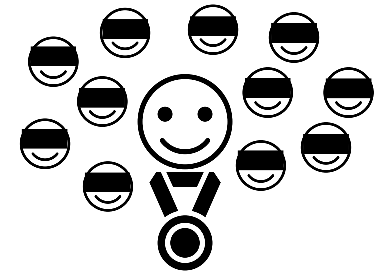
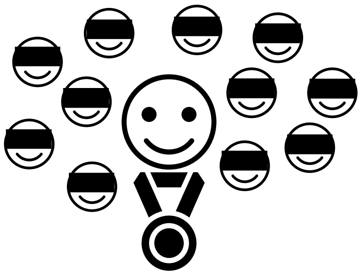
Intra-task jumps

Results - Selection



Time	Advice
$> \; diam$	0
$a \cdot diam,$ $a \in (0, 1)$	$\Theta(\log \log diam)$
$diam^e,$ $e < 1$	$\Theta(\log diam)$

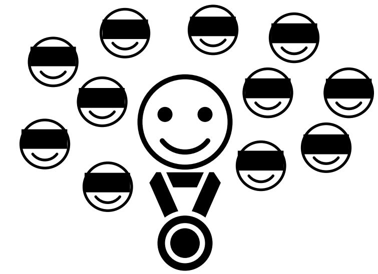
Results - Selection



Time	Advice
$> \ diam$	0
$a \cdot diam,$ $a \in (0,1)$	$\Theta(\log\log diam)$
$diam^e,$ $e < 1$	$\Theta(\log diam)$

Only valid for rings

Results - Selection



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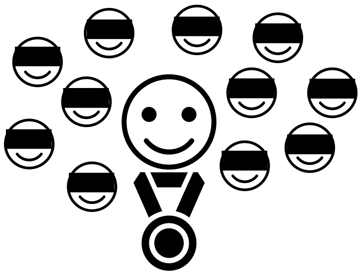
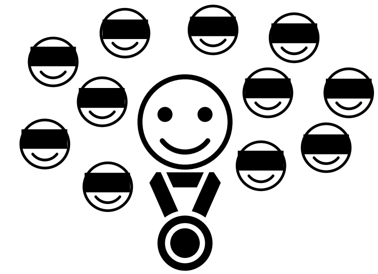


Time	Advice
$> \ diam$	0
$a \cdot diam$, $a \in (0,1)$	$\Theta(\log\log diam)$
$diam^e$, $e < 1$	$\Theta(\log diam)$

We will go through the algorithm for the upper bound

Results - Selection



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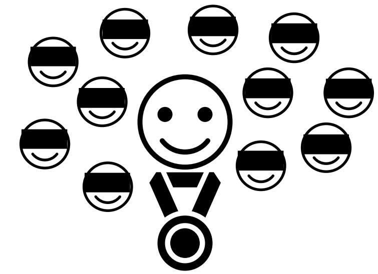
$> \ diam$	0
$a \cdot diam,$ $a \in (0,1)$	$\Theta(\log\log diam)$
$diam^e,$ $e < 1$	$\Theta(\log diam)$

For rings $\Theta(\log diam) =$
$\Theta(\log n)$

Results - Selection



Time	Advice
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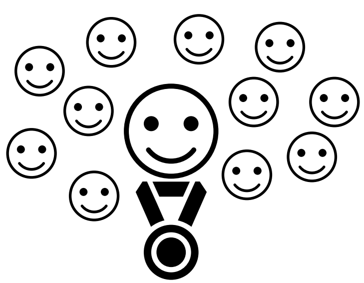
$a \cdot diam$, $a \in (0,1)$	$\Theta(\log\log diam)$
$diam^e$, $e < 1$	$\Theta(\log diam)$

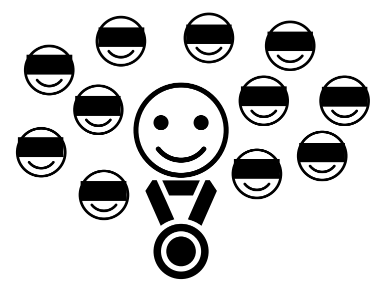
Intra-task jumps

Results

Time	Advice
$> \ diam$	0
$diam$	$\Theta(\log diam)$
$< \ diam$	$\Theta(\log n)$

Time	Advice
$> \ diam$	0
$a \cdot diam,$ $a \in (0,1)$	$\Theta(\log \log diam)$
$diam^e,$ $e < 1$	$\Theta(\log diam)$



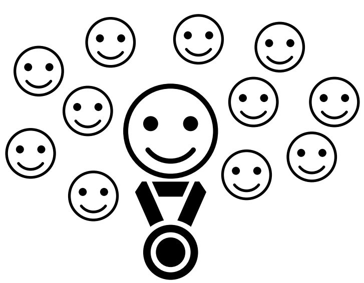


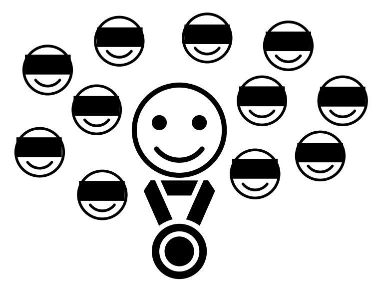
Results

Inter-task jump

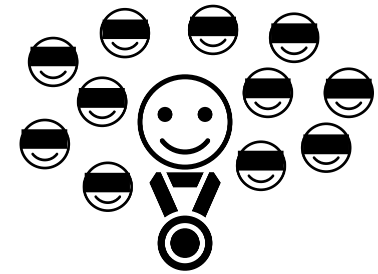
Time	Advice
$> \ diam$	0
$diam$	$\Theta(\log diam)$
$< \ diam$	$\Theta(\log n)$

Time	Advice
$> \ diam$	0
$a \cdot diam$, $a \in (0, 1)$	$\Theta(\log \log diam)$
$diam^e$, $e < 1$	$\Theta(\log diam)$





Selection - Example

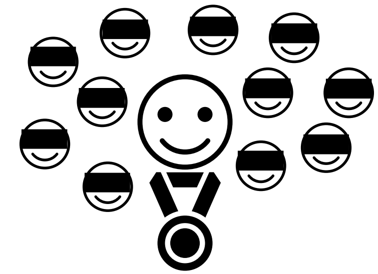


Time constraint: t $= \ a \cdot diam\ , a \in (0,1)$

Goal : Prove that size of advice is $\mathrm{O}(\log\log diam(R))$ for any ring R

Simplification: $a = 1$, so $\ t = diam$

Selection - Example



Algorithm consists of two stages

1. Round-by-round discovery
2. Eliminate resulting nodes from first stage

Advice string split in two parts $A = A_1 A_2$

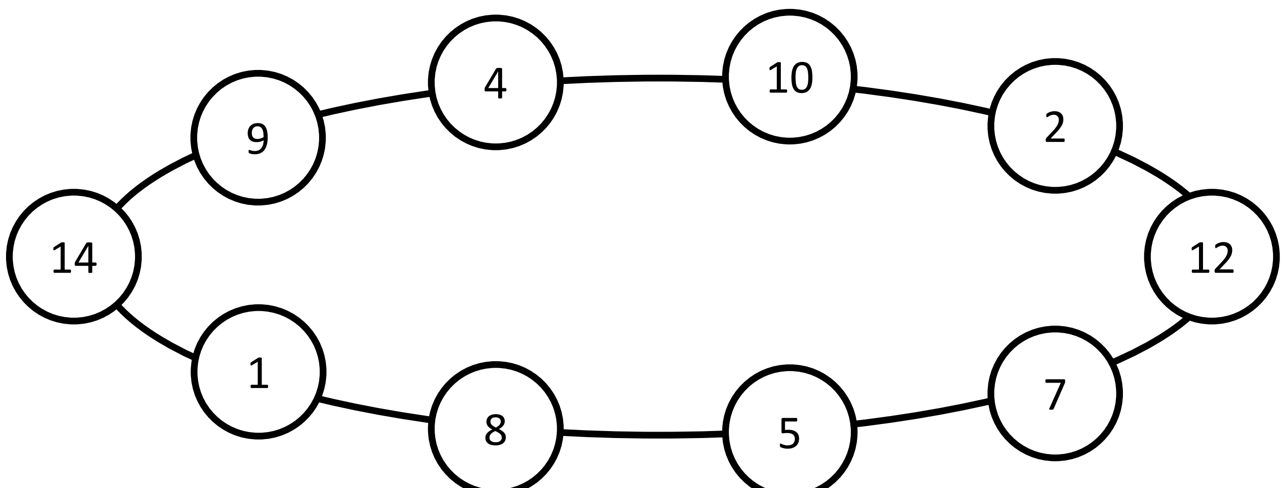
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$



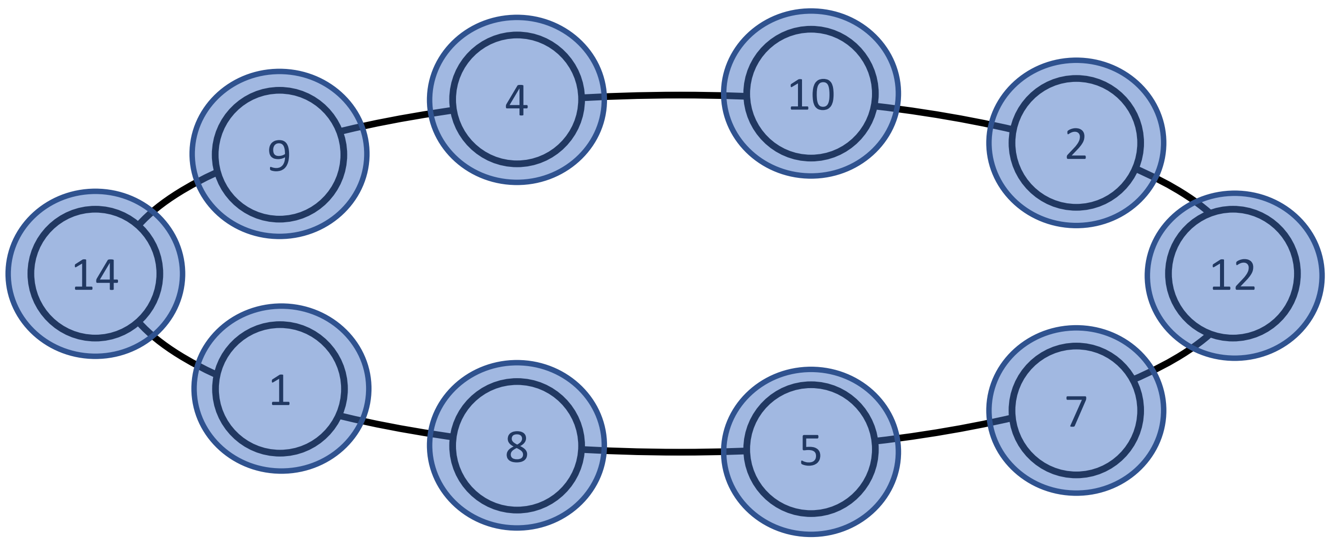
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$



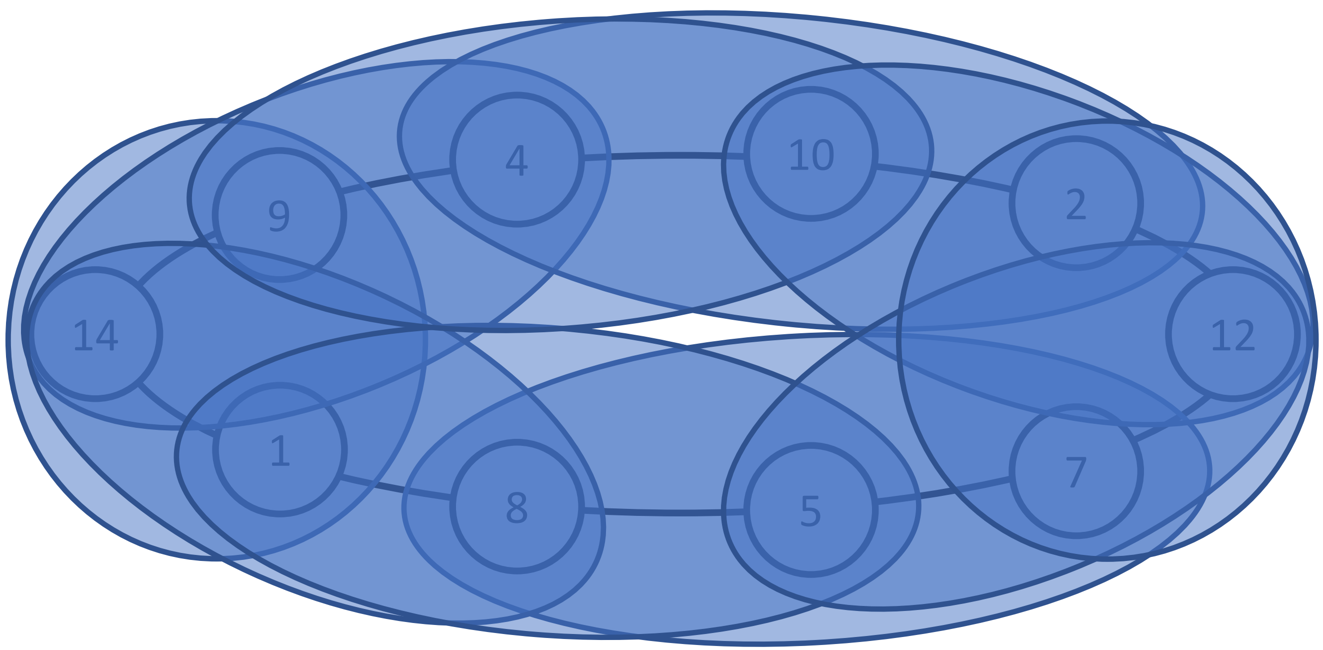
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$



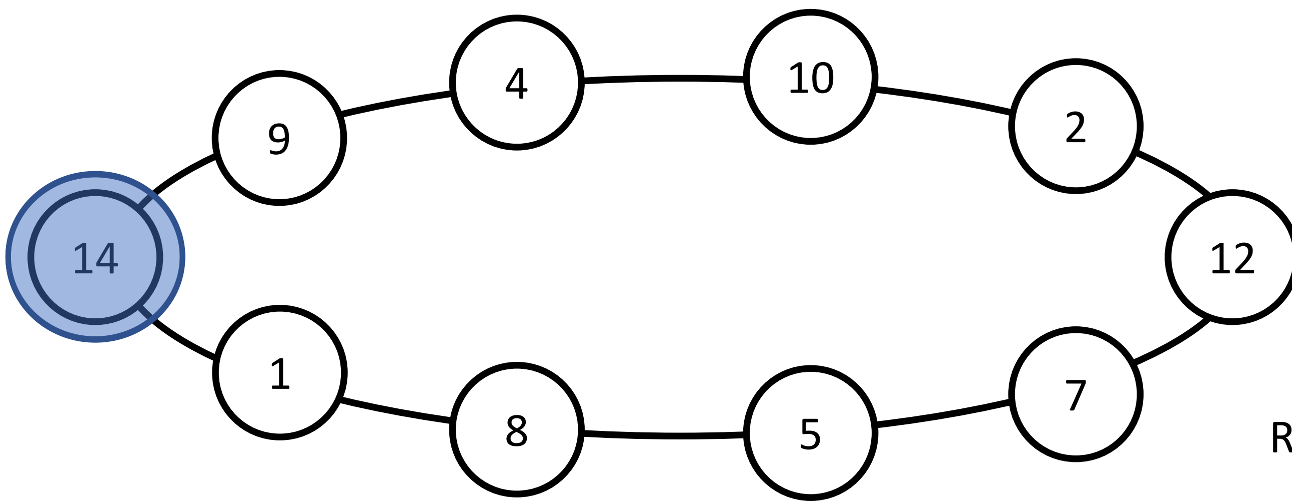
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$



Round 0

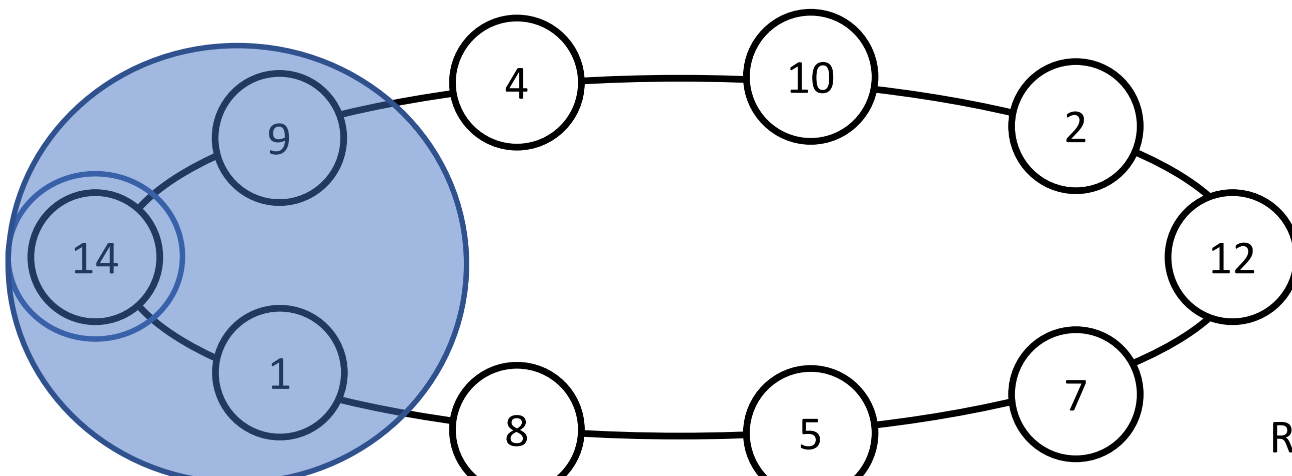
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\qquad$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $\qquad$ $r = 4$



Round 1

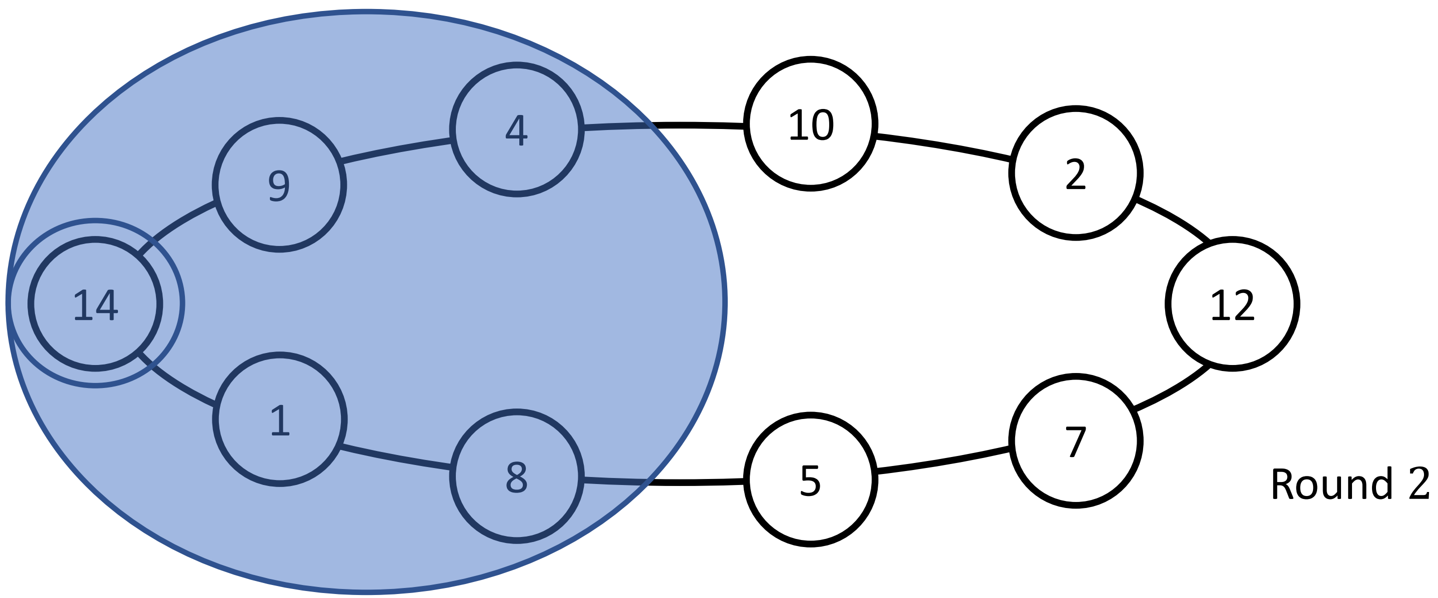
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$



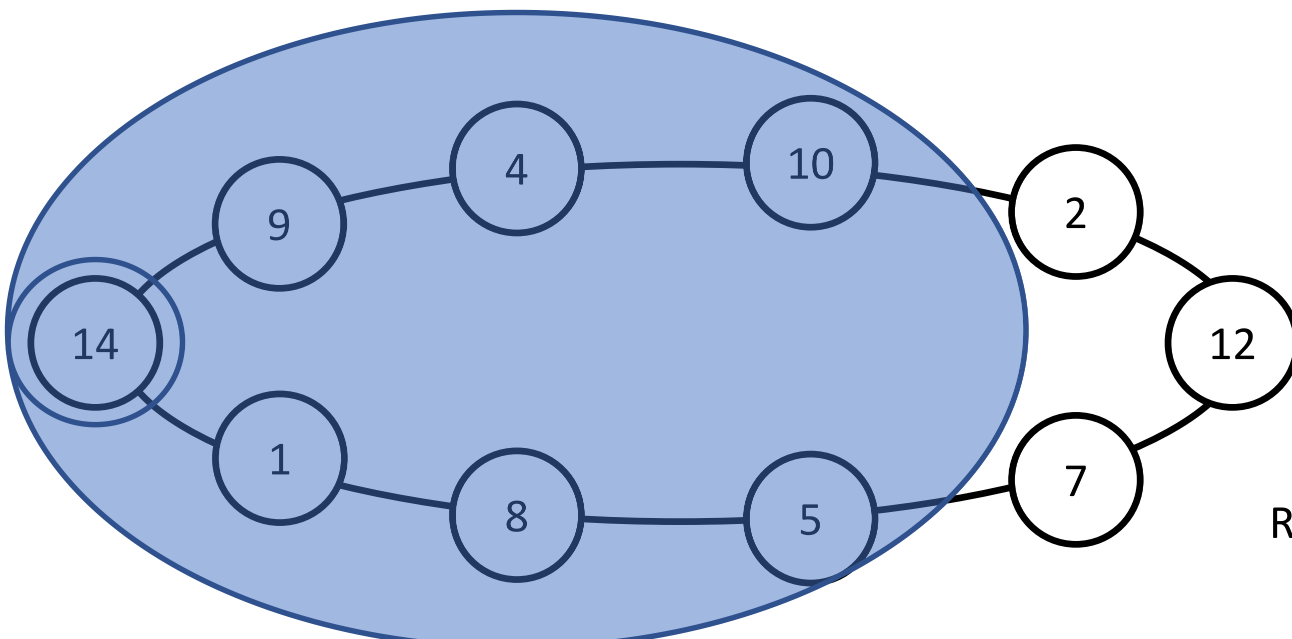
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$



Round 3

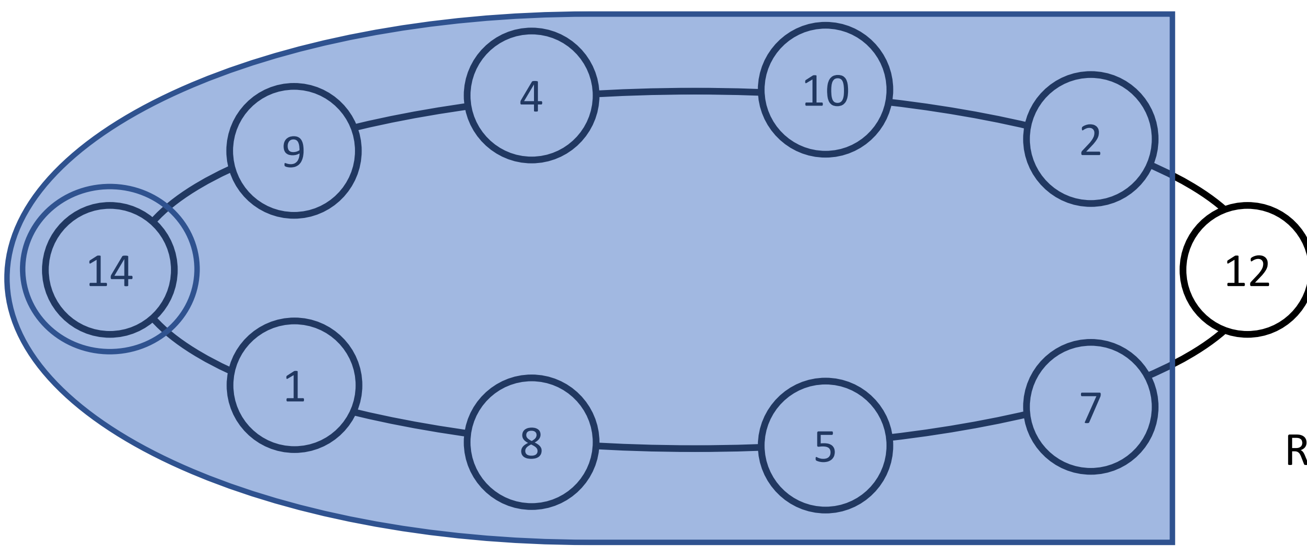
Selection – Example

Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$



Round 4

Selection – Example

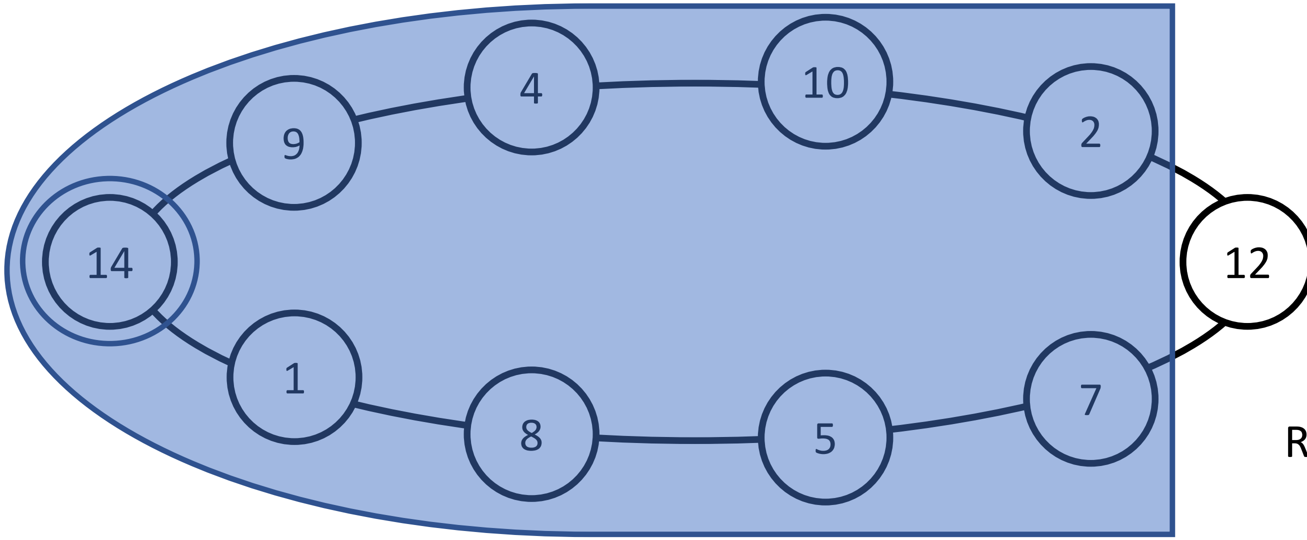
Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$

If $v \neq \max(\Lambda(r, v))$ output 0



Round 4

Selection – Example

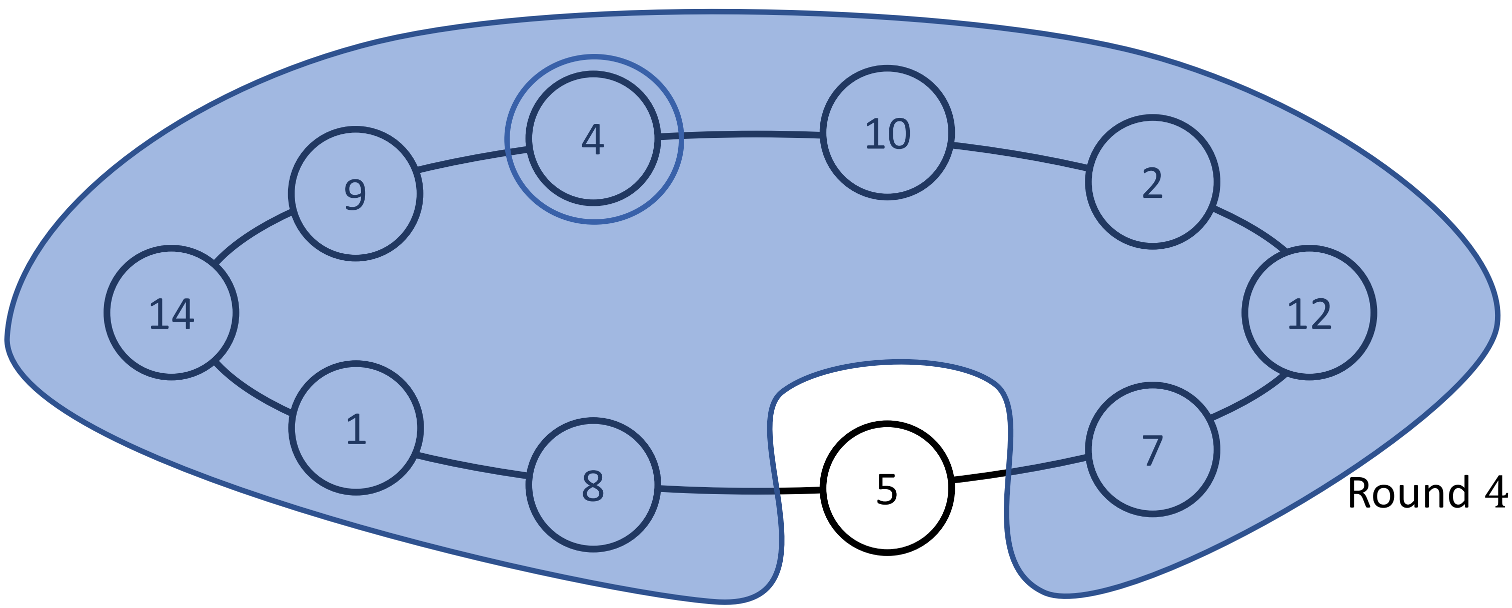
Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$

If $v \neq \max(\Lambda(r, v))$ output 0



Selection – Example

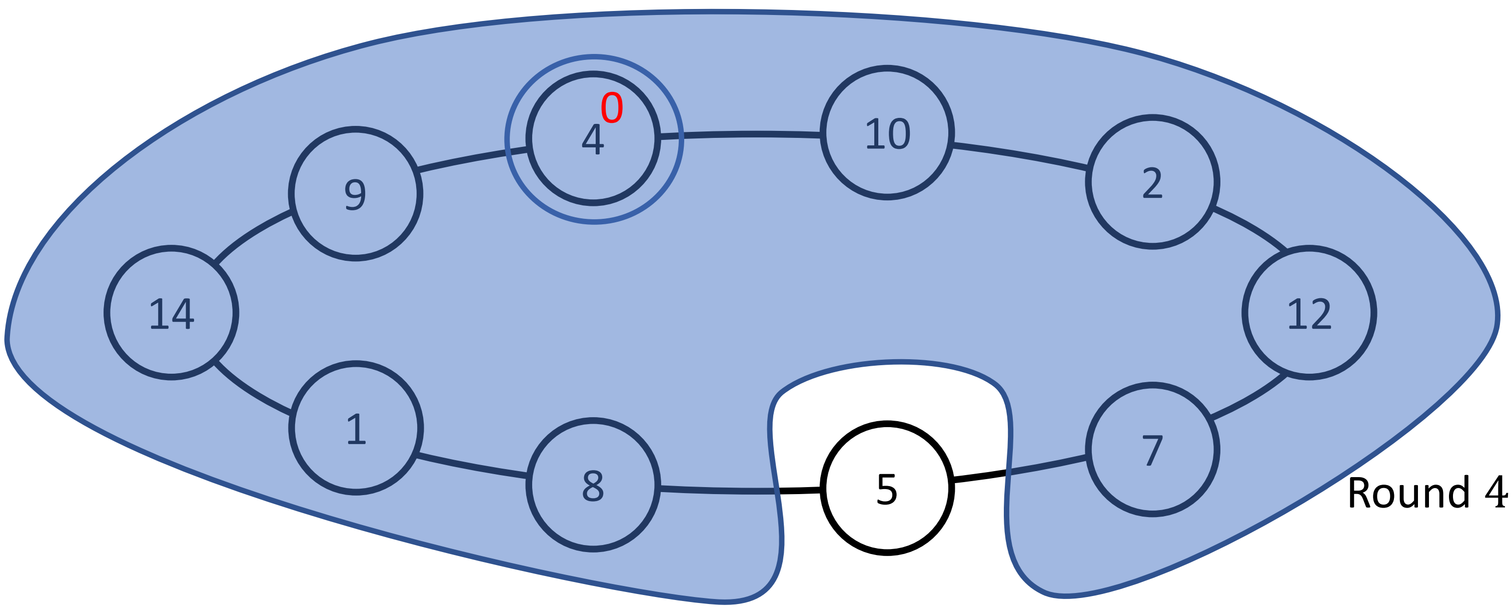
Algorithm - Stage 1

$A_1 = \lfloor \log(diam(R)) \rfloor$ $\lfloor \log(5) \rfloor = 2$

On each node v

Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$ $r = 4$

If $v \neq \max(\Lambda(r, v))$ output 0



Selection – Example

Algorithm - Stage 2: Advice construction

$C_R = \{\gamma_0, \gamma_1, \ldots, \gamma_{|C_R|-1}\}$ = set of resulting nodes where γ_0 is largest

Goal: eliminate all but γ_0

Selection – Example

Algorithm - Stage 2: Advice construction

$C_R = \{\gamma_0, \gamma_1, \ldots, \gamma_{|C_R|-1}\}$ = set of resulting nodes where γ_0 is largest

Goal: eliminate all but γ_0

Solution: for each $\gamma_{j,j>0}$, find difference with γ_0 and provide it as advice

Selection – Example

Algorithm - Stage 2: Advice construction

$C_R = \{\gamma_0, \gamma_1, \dots, \gamma_{|C_R|-1}\}$ = resulting set of nodes where γ_0 is largest

Goal: eliminate all but γ_0

Solution: for each $\gamma_{j,j>0}$, find difference with γ_0 and provide it as advice

$$\gamma_0 = 100111$$

$$\gamma_1 = 100011$$

Selection – Example

Algorithm - Stage 2: Advice construction

$C_R = \{\gamma_0, \gamma_1, \ldots, \gamma_{|C_R|-1}\}$ = resulting set of nodes where γ_0 is largest

Goal: eliminate all but γ_0

Solution: for each $\gamma_{j, j>0}$, find difference with γ_0 and provide it as advice

$$\gamma_0 = 100111$$

$$\gamma_2 = 100101$$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \emptyset$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \emptyset$

$$\gamma_0 = 100111$$

$$\gamma_1 = 100011$$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \{3\}$

$$\gamma_0 = 100111$$

$$\gamma_1 = 100011$$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \{3\}$

$$\gamma_0 = 100111$$

$$\gamma_2 = 100101$$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \{3, 4\}$

$$\gamma_0 = 100111$$

$$\gamma_2 = 100101$$

Selection – Example

Algorithm - Stage 2: Advice construction

A_2 set of indices satisfying:

For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

Construction Example

$A_2 = \{3, 4\}$

$$\gamma_0 = 100111$$

$$\gamma_3 = 100001$$

Selection – Example

Algorithm - Stage 2: Algorithm

On each node γ in $\mathrm{C_R}$

If there exists $\mathrm{i} \in A_2$ such that $\gamma[i] = 0$

Output 0

Else

Output 1

Selection – Example

Algorithm - Stage 2: Algorithm

On each node γ in $\mathrm{C_R}$

If there exists $\mathrm{i} \in A_2$ such that $\gamma[i] = 0$

Output 0

Else

Output 1

Algorithm at $\boldsymbol{\gamma_2}$

$A_2 = \{3, 4\}$

$$\gamma_2 = 100101$$

Selection – Example

Algorithm - Stage 2: Algorithm

On each node γ in $\mathrm{C_R}$

If there exists $\mathrm{i} \in A_2$ such that $\gamma[i] = 0$

Output 0

Else

Output 1

Algorithm at $\boldsymbol{\gamma_2}$

$$A_2 = \{3, 4\}$$

$$\gamma_2 = 100101$$

=> Output 0

Selection – Example

Algorithm - Recap

$A = A_1 A_2$

$A_1 = \lfloor \log(diam(R)) \rfloor$
A_2 : For all $\gamma_{j,j>0}$, there exists $i \in A_2$ such that $\gamma_j[i] = 0$ and $\gamma_0[i] = 1$

On each node v
    Run $r = 2^{A_1}$ rounds to learn $\Lambda(r, v)$
    If $v \neq \max(\Lambda(r, v))$ output 0

On each node γ in $\mathrm{C_R}$
    If there exists i $\in A_2$ such that $\gamma[i] = 0$
        Output 0
    Else
        Output 1

Selection – Example

Algorithm - Size of Advice

$A_1 = \lfloor \log diam(R) \rfloor$

Size of A_1 is $\mathrm{O}\big(\log \log diam(R)\big)$

$A_2 =$ is a set of at most $|C_R|$ indices

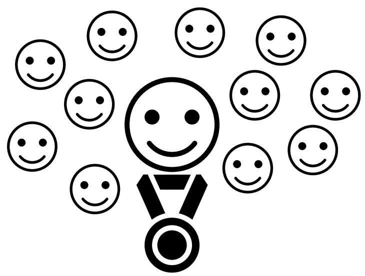
Size of each index is $\mathrm{O}\big(\log \log diam(R)\big)$
$|C_R|$ is a constant

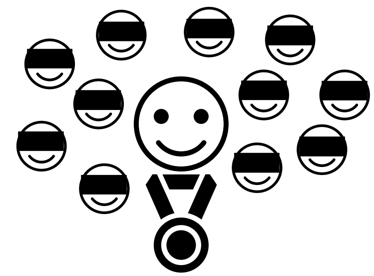
Size of A_2 is $\mathrm{O}\big(|C_R| \cdot \log \log diam(R)\big) = \mathrm{O}\big(\log \log diam(R)\big)$

Size of advice A is $\mathrm{O}\big(\log \log diam(R)\big)$

Note that the time constraint is also respected.

Summary





- Election vs Selection
- Algorithm with advice
- Measure of difficulty – size of advice
- Results
- Algorithm overview for selection in time linear in the diameter (upper bound)

Questions ?

Related Work

- Message complexity
- Non-unique labels
- Non-labelled graphs
- Election of arbitrary node
- Algorithms with advice for other problems
- Different advice for each node
- Different kind of difficulty measurement.