Exercise 12

Lecturer: Mohsen Ghaffari

1 Pipelining

We consider an arbitrary n-node network $G = (V, E)$ with diameter D. Moreover, we work in the $\mathsf{CONGEST}$ model of distributed computing where each node has an $O(\log n)$-bit unique identifier and per round, each node can send $O(\log n)$ bits to each of its neighbors.

Exercises

(1a) Suppose that each node $v \in V$ is given k different inputs $x_1(v)$, $x_2(v)$, ..., $x_k(v)$, each being a $\Theta(\log n)$-bit number. The objective is to for all nodes to know the outputs $y_i = \min_{v \in V} x_i(v)$, for each $i \in \{1, 2, \ldots, k\}$. Devise a deterministic distributed algorithm for this problem with round complexity $O(D + k)$.

(2a) Suppose there are k messages m_1, m_2, ..., m_k, each initially placed at an arbitrary node of the network (many or even all of the messages may be placed on the same node). Consider the following basic algorithm: per round, each node v picks one of the messages m_i that it has (from the beginning or received in the past) and send m_i it to all of its neighbors; node v will never send m_i again. Notice that a node will not send two of the messages at the same time. Prove that if we run this algorithm for $O(D + k)$ rounds, all nodes will receive all the messages.

2 Minimum Spanning Tree

Consider an undirected connected graph $G = (V, E)$ where $n = |V|$. Suppose that each node $v \in V$ has selected one of its incident edges (v, u) as the *proposal edge of* v, let us denote it $e_v = (v, u)$. For instance, in the MST algorithm of Boruvka, this would be the minimum-weight edge incident on v. Notice that the two endpoints of an edge might propose this one edge simultaneously. Consider the random process that each node flips a fair coin for itself and then, we mark the proposed edge $e_v = (v, u)$ of node v only if v draws tail and u draws head.

Exercises

(2a) Prove that, in expectation, we mark at least $n/8$ edges.

(2b) Prove that, if we contract all the marked edges, the resulting graph has at most $7n/8$ nodes, in expectation.

(2c) Consider repeating the above process for $20 \log n$ iterations: In each iteration, we contract all the marked edges, and maintain only the "outgoing edges", i.e., those edges that have exactly one endpoint in this contraction. Then, select one min-weight outgoing edge per new node, and repeat the marking process as above using one coin toss per each new node. Use (2b) to prove that, after $20 \log n$ iterations, with high probability, we have contracted everything to a single node.