Cheap Labor Can Be Expensive

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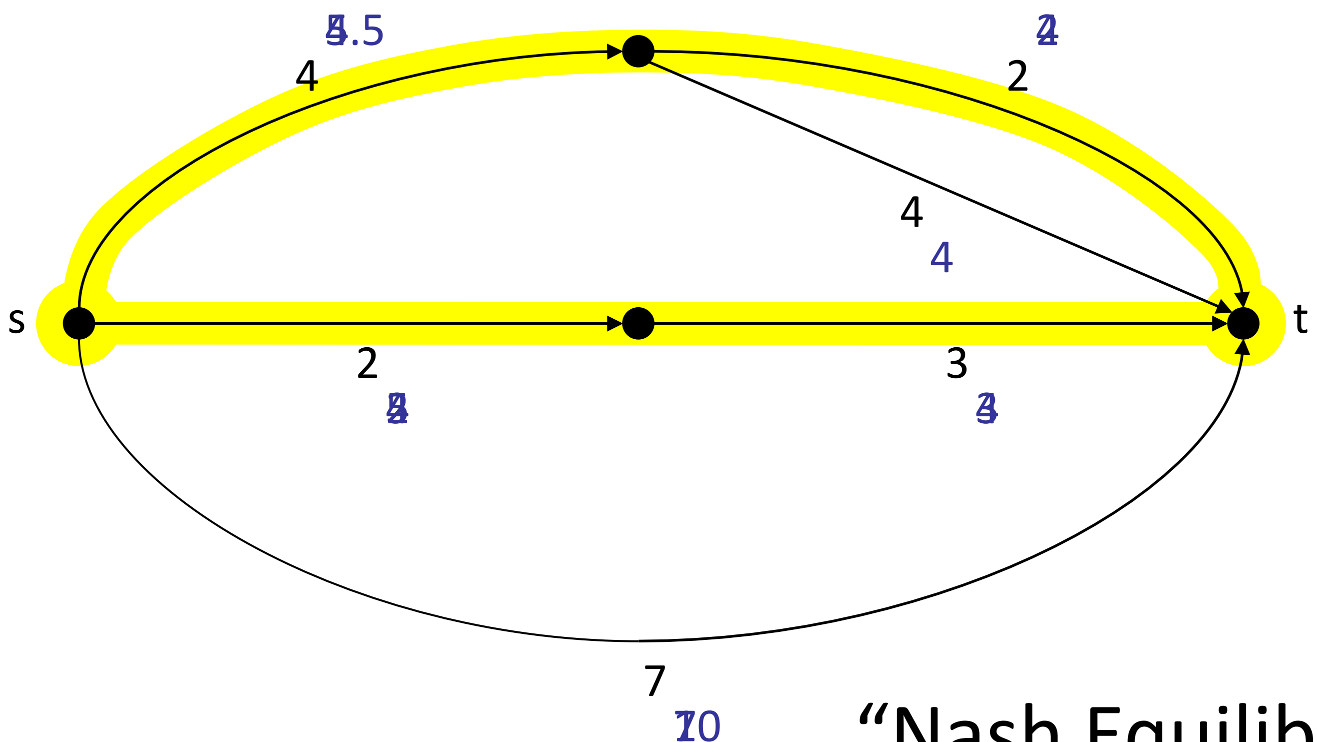
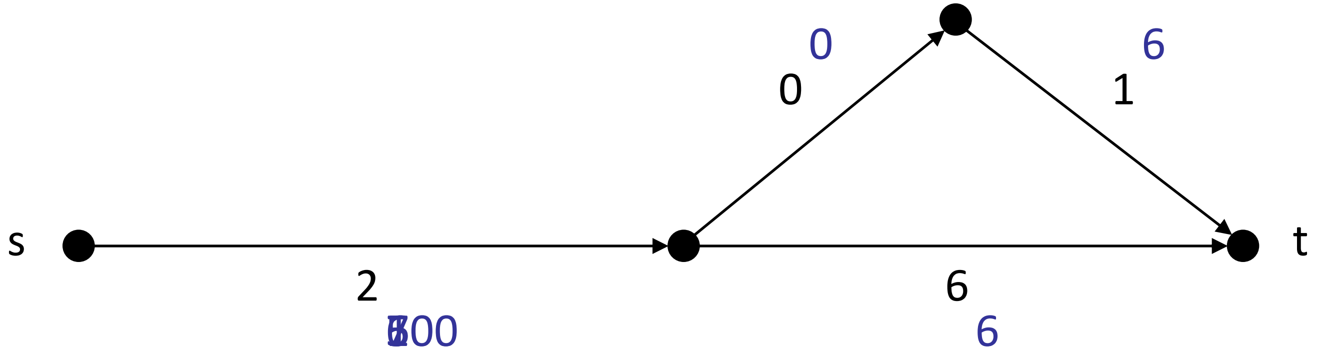
Michael König

The Problem

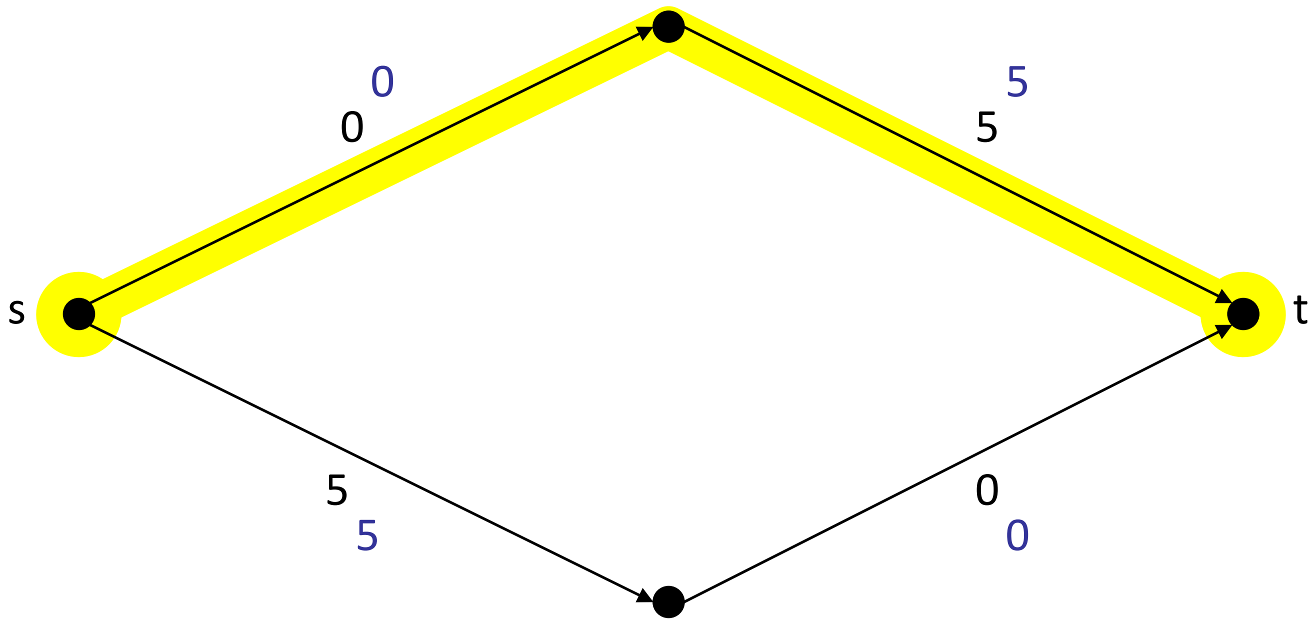


“Nash Equilibrium”

The Problem

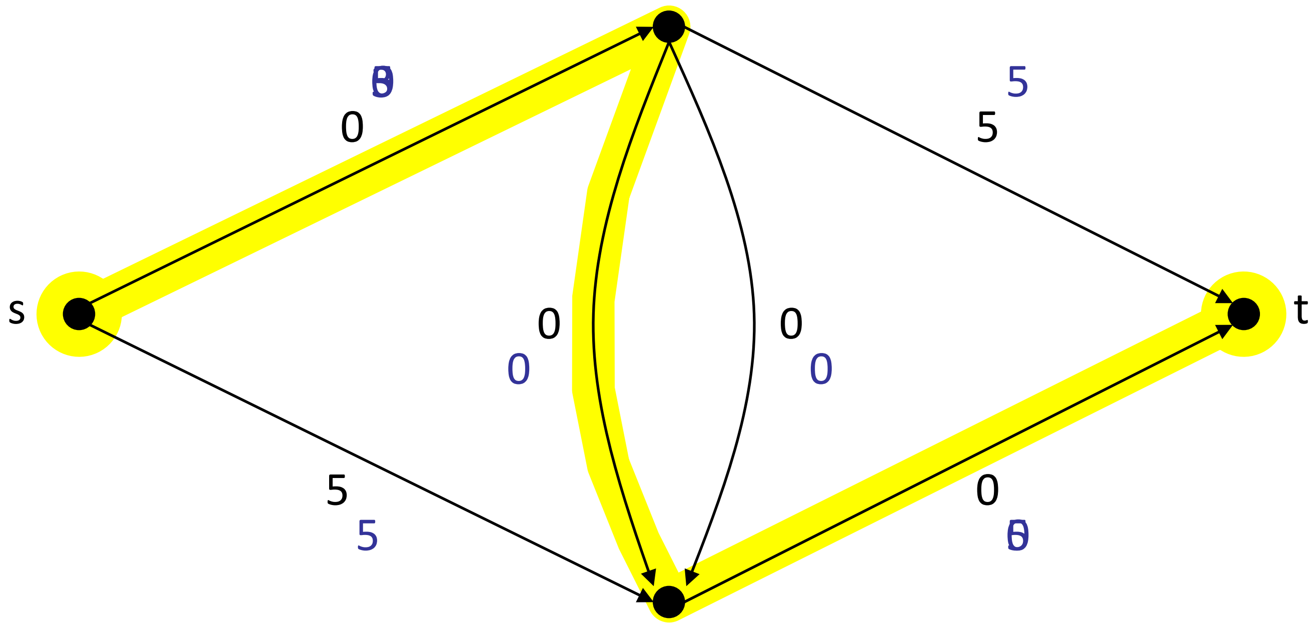


The Problem



Total price: 5

The Problem

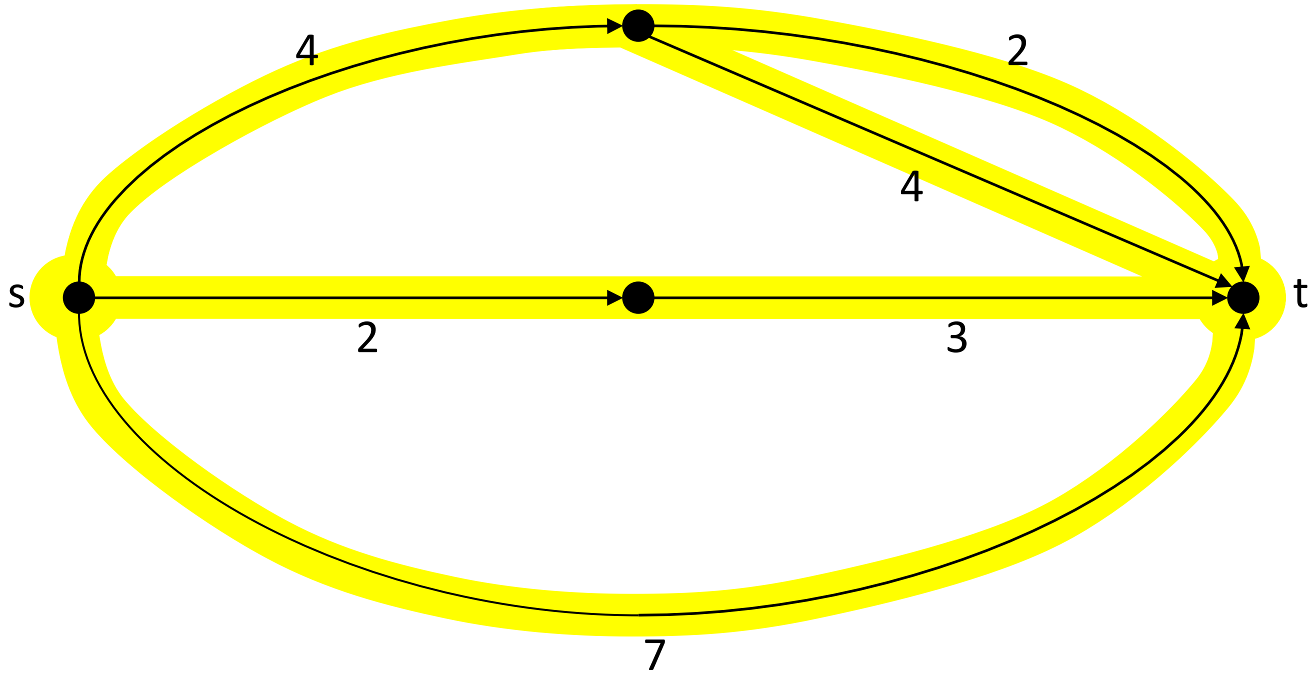


Total price: 10

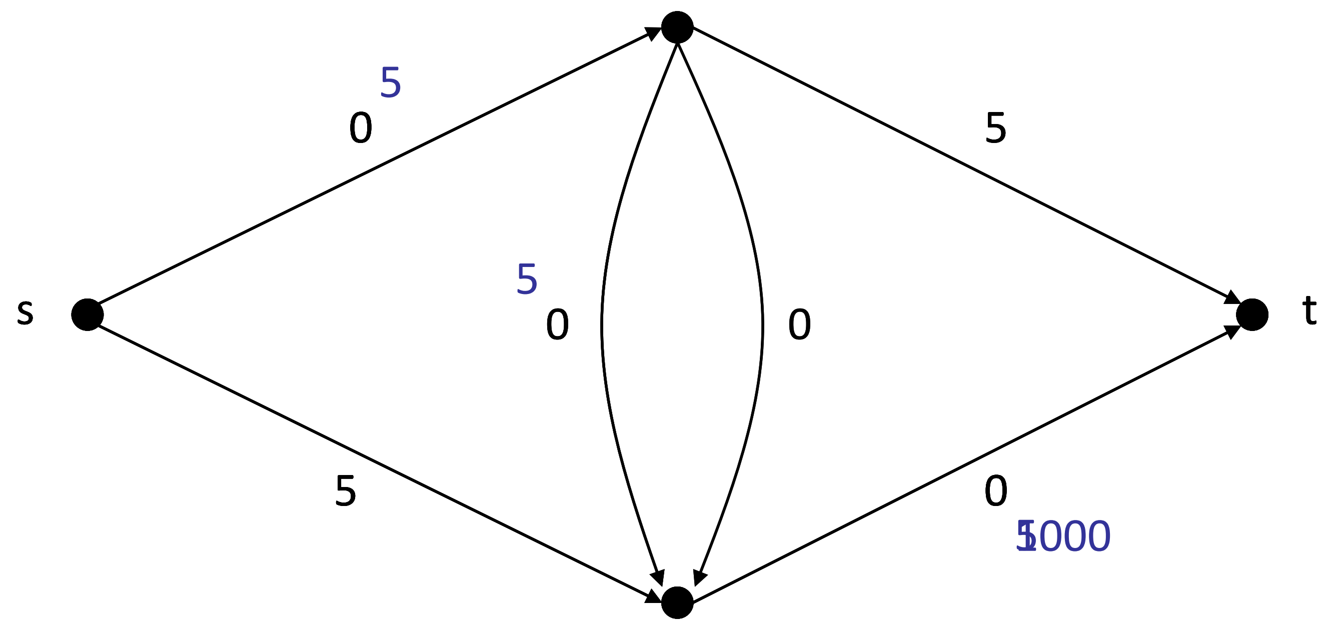
Markets

- Set of agents "E"
- Each agent $e \in E$ has a **cost c(e)** and **bid b(e)**
- Customer wants to hire a team of agents
- **Feasible sets** "F" are teams of agents capable of getting the job done

Feasible Sets



Cheap Labor Cost



Total price: 10

→ Cheap Labor Cost of this market is $\frac{10}{5} = 2$

Cheap Labor Cost

- Cheap labor cost for a market M:

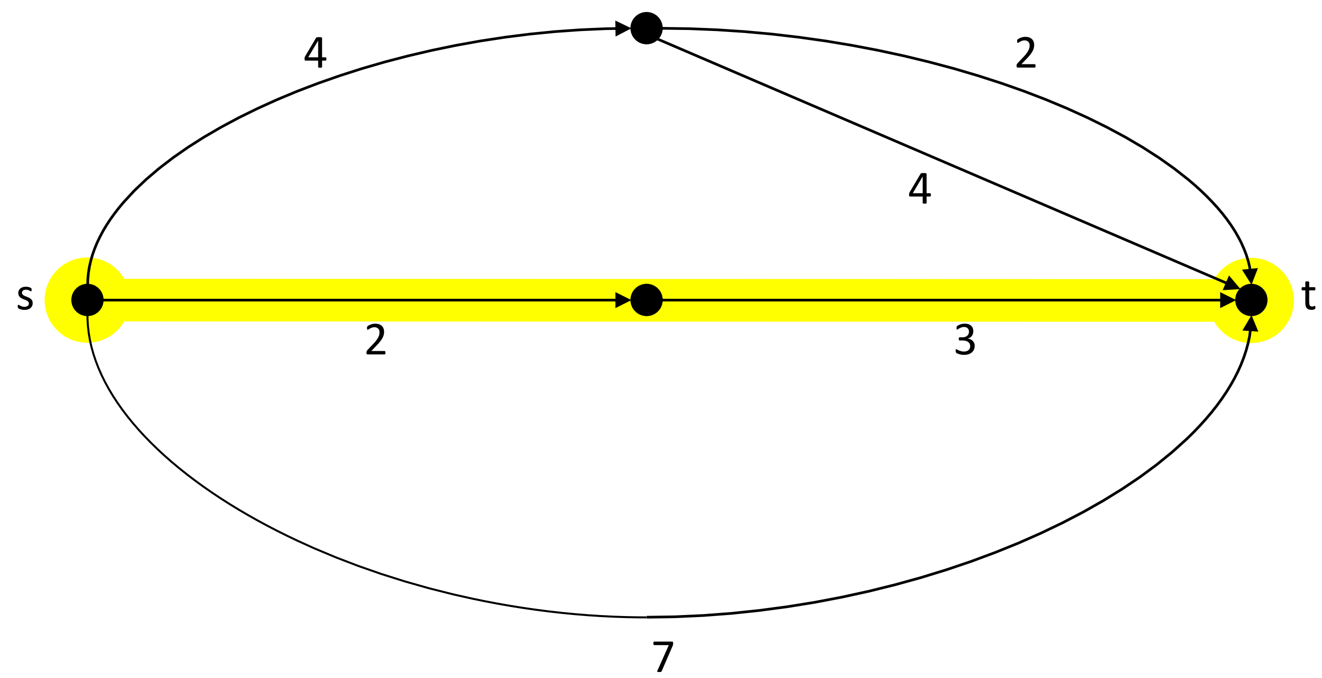
$$\frac{p_M}{\min_{M' \subseteq M} p_{M'}}$$

- p_M := total price of M

Questions up to this point?

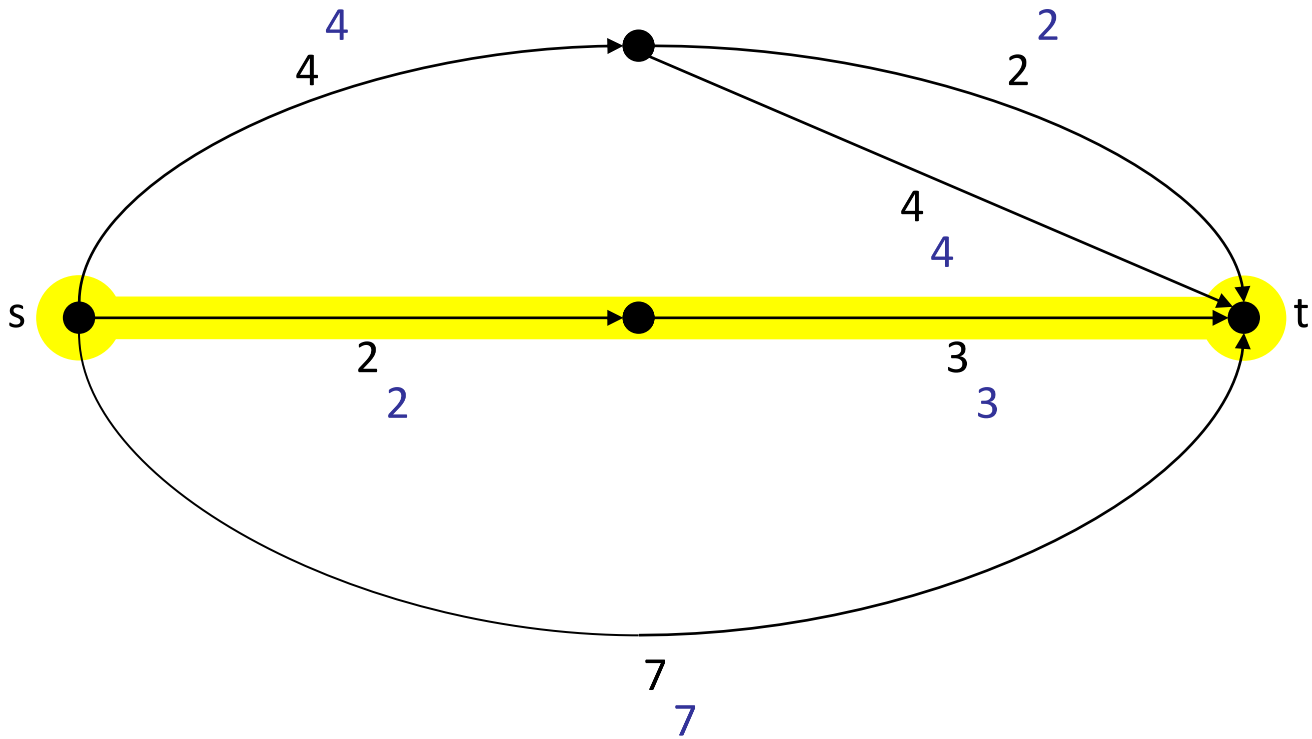
GreedyAlg

1. Find the cheapest feasible set S ∈ F with respect to costs



GreedyAlg

2. For each $e \in E$, initialize $b(e)$ to $c(e)$

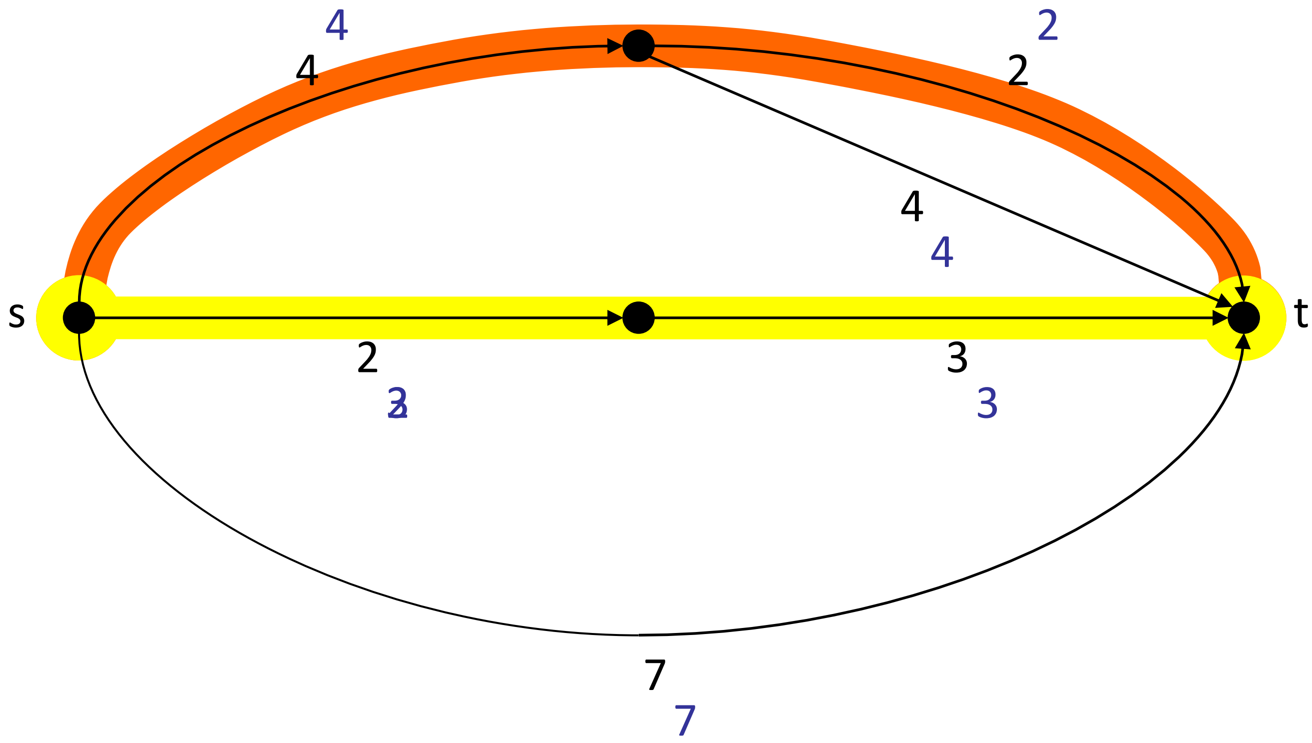


GreedyAlg

3. For each $e \in S$:

   - Raise $b(e)$ until there is $S' \in F$ such that $e \notin S'$ and $b(S) = b(S')$

$$b(S) = \sum_{e \in S} b(e)$$



GreedyAlg

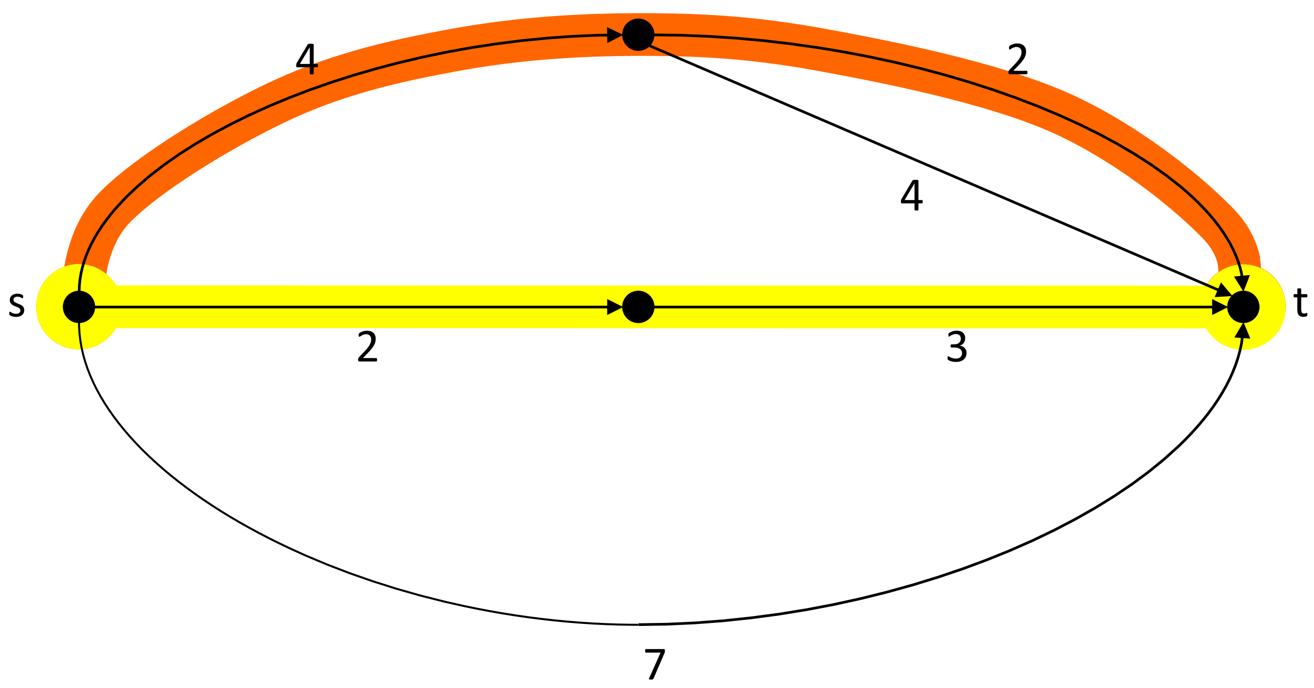
1. Find the cheapest feasible set $S \in F$ with respect to costs
2. For each $e \in E$, initialize $b(e)$ to $c(e)$
3. For each $e \in S$:

   - Raise $b(e)$ until there is $S' \in F$ such that $e \notin S'$ and $b(S) = b(S')$

4. Output the bids b and the winning set S

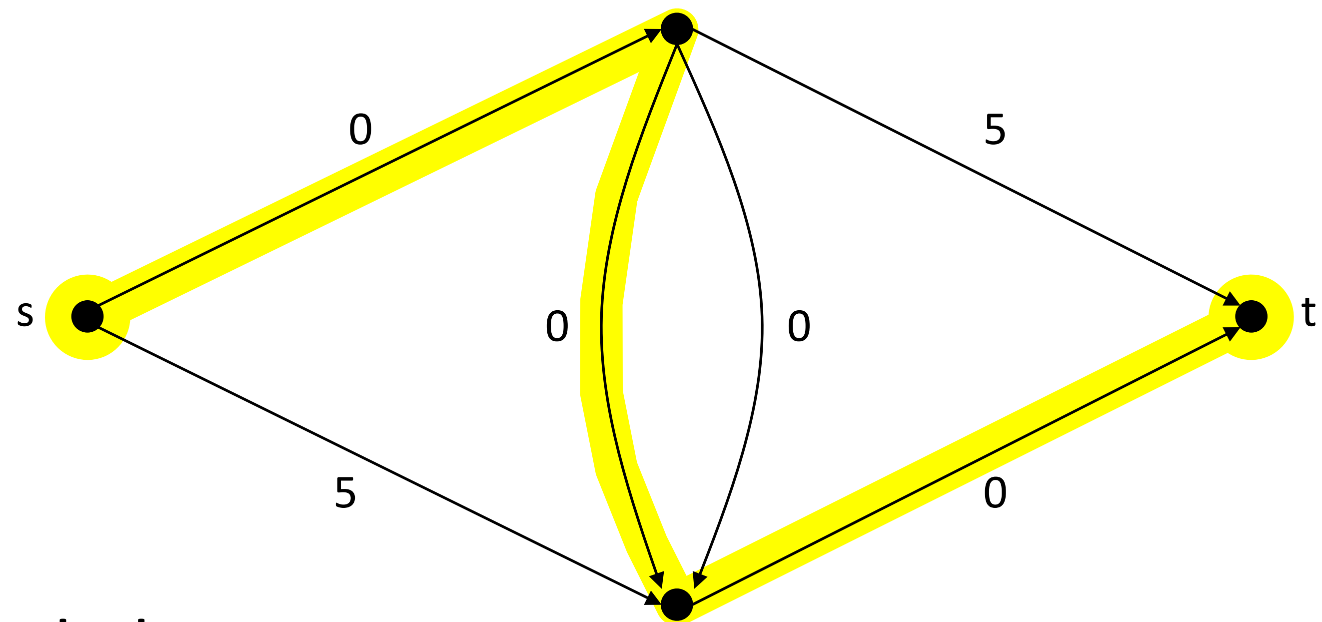
Tight sets

- For any NE b with winning set S:
  - For any $e \in S$, there is another winning feasible set $S' \in F$ with $e \notin S'$ and $b(S) = b(S')$
  - These feasible sets are called **tight sets**.



Upper Bound

- The cheap labor cost of any market is at most |S|, where $S \in F$ is a feasible set with minimum total cost



Here: |S| = 3

Proof of Upper Bound

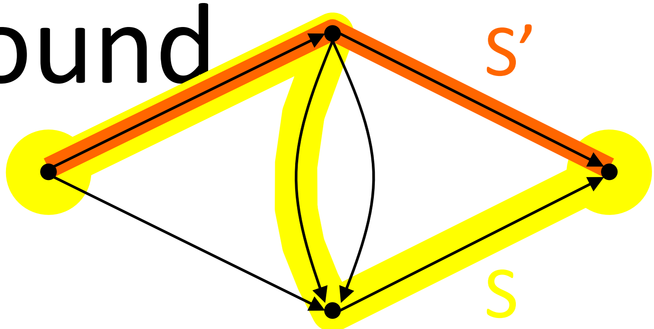
- It suffices to show:
  - For any market M, NE b with winning set S, for any submarket M', best NE b' with winning set S'

$$b(S) \leq |S| \cdot b'(S')$$

$$\frac{b(S)}{b'(S')} \leq |S|$$

(we choose b and S to be computed by `GreedyAlg`)

Proof of Upper Bound



Case 1: e ∈ S’ \ S

- b(e) = c(e)

→ b(S’ \ S) = c(S’ \ S)

[GreedyAlg]

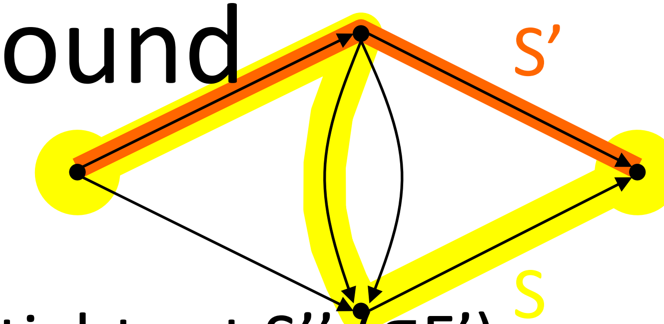
- b(S \ S’) ≤ b(S’ \ S)
- c(S’ \ S) ≤ b’(S’ \ S)

[S is the winning set]

[bid behavior]

→ b(S \ S’) ≤ b’(S’ \ S)

Proof of Upper Bound



Case 2: $e \in S' \cap S$

- For each such e there exists a tight set S'' ($\in F'$) such that $e \notin S''$ and $b'(S') = b'(S'')$.
- We claim $b(e) \leq b'(S')$. Otherwise:

$$
\begin{aligned}
b(S) &= b(S \setminus S'') + b(S \cap S'') \\
&> b'(S') + b(S \cap S'') && \text{[reverse claim]} \\
&= b'(S'') + b(S \cap S'') && \text{[}b'(S') = b'(S'')\text{]} \\
&\geq c(S'') + b(S \cap S'') && \text{[bid behavior]} \\
&\geq c(S'' \setminus S) + b(S \cap S'') \\
&= b(S'' \setminus S) + b(S \cap S'') && \text{[GreedyAlg]} \\
&= b(S'') && \text{[contradiction: S is the winning set]}
\end{aligned}
$$

Proof of Upper Bound

Case 1 ($e \in S' \setminus S$): $b(S \setminus S') \leq b'(S' \setminus S)$

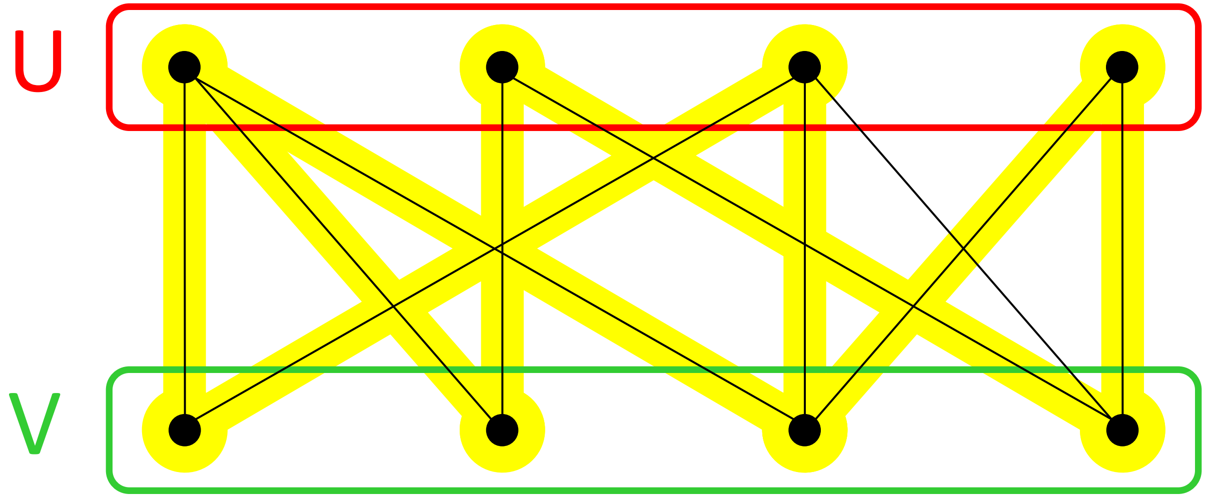
Case 2 ($e \in S' \cap S$): $b(e) \leq b'(S')$

Putting the cases together:

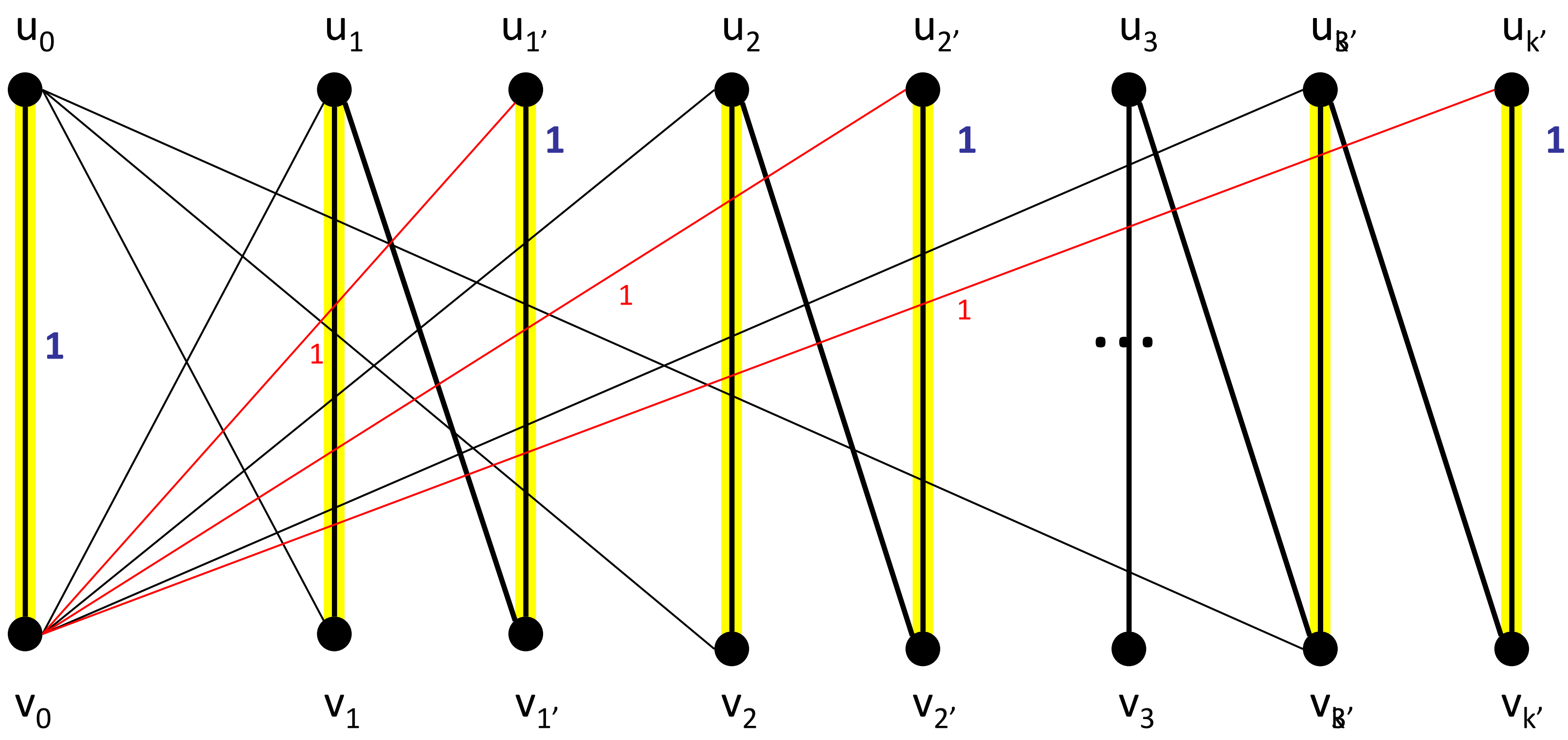
$$
\begin{aligned}
b(S) &= b(S \setminus S') + b(S \cap S') \\
&\leq b'(S' \setminus S) + |S \cap S'| \cdot b'(S') \\
&\leq |S| \cdot b'(S')
\end{aligned}
$$

Perfect Bipartite Matching Markets

- Customer wants to buy edges to obtain a perfect matching in a bipartite graph

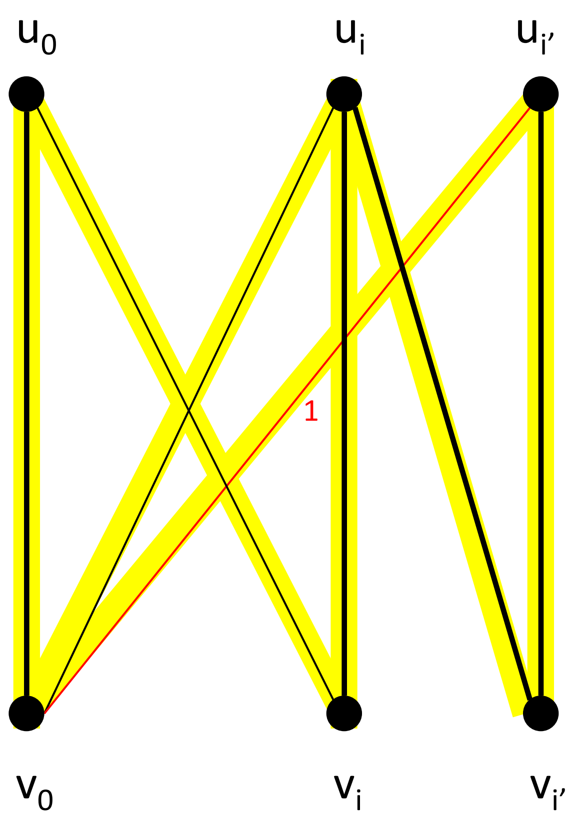


Perfect Bipartite Matching Markets



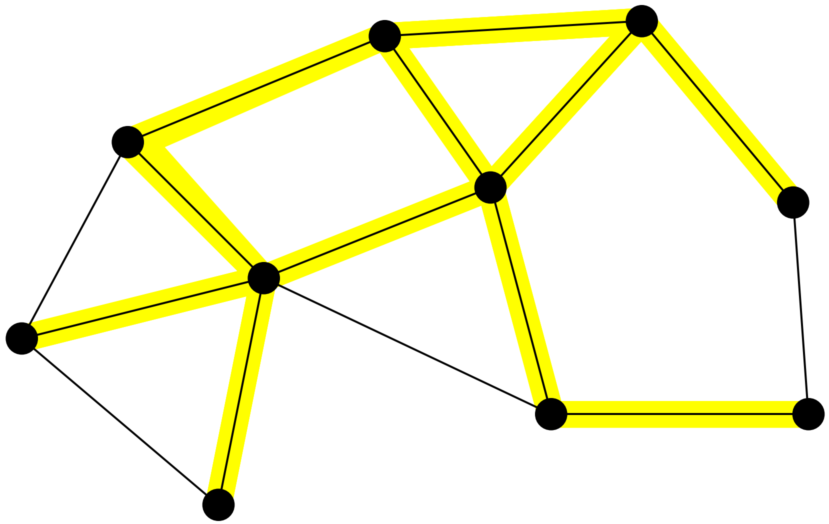
$p_M = k$ $p_{M'} = 1$ Cheap labor cost = k = O(|V|)

Perfect Bipartite Matching Markets



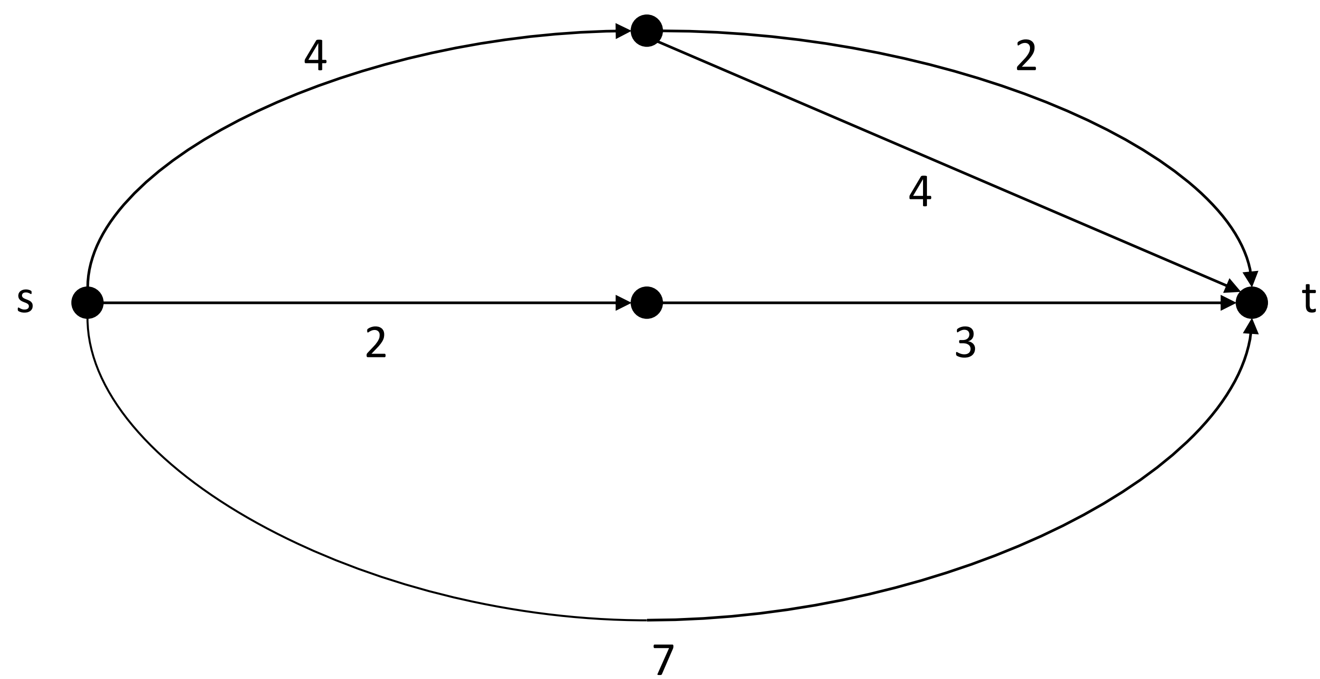
Matroid Markets

- Agents and feasible sets form a matroid (E, F) (F ⊆ **P**(E) with a bunch of special rules)
- Cheap labor cost is always 1.
- Natural Occurrence: buying spanning trees



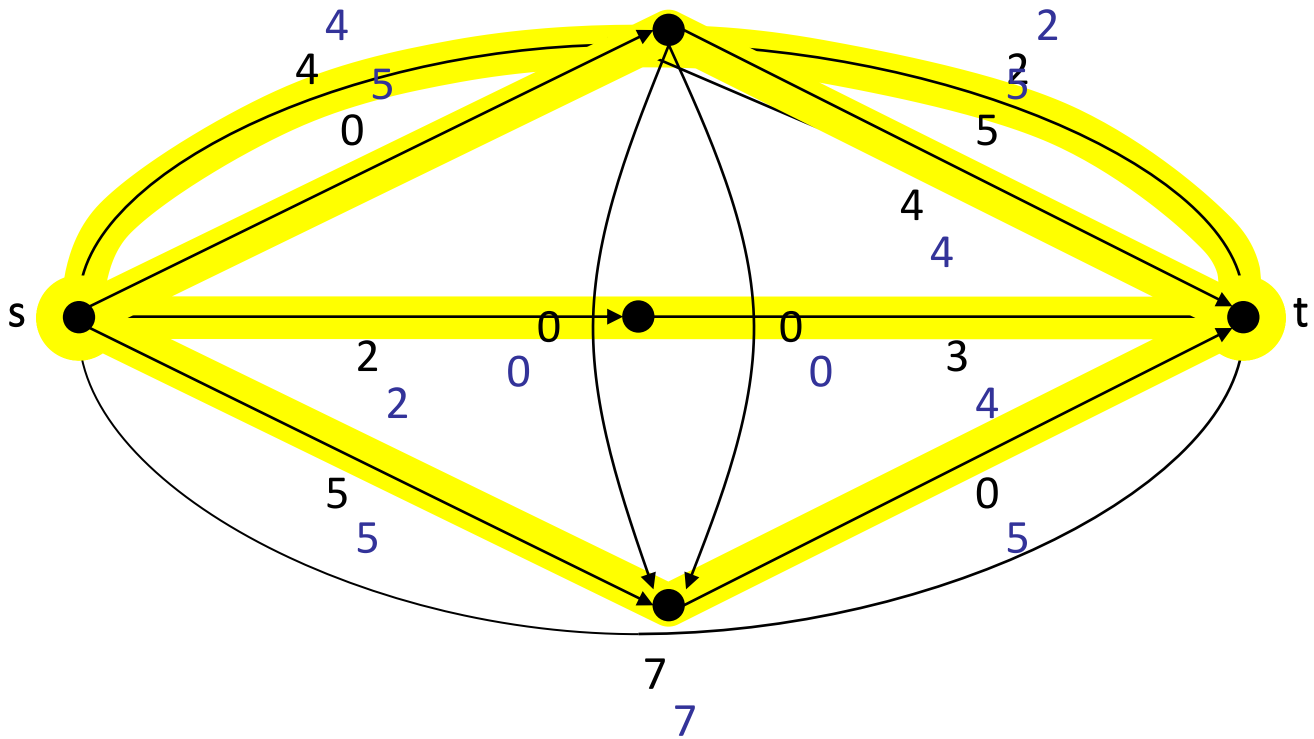
Path Markets

- Purchasing an s-t path in a directed graph

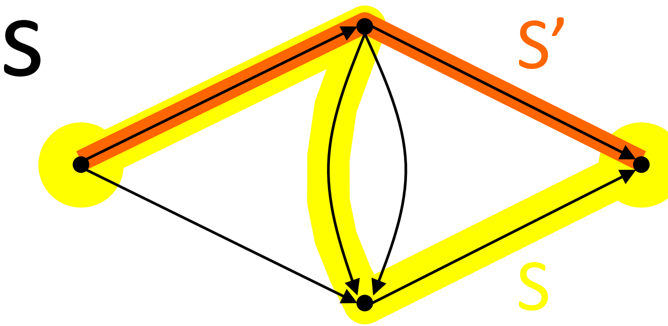


Path Markets

- Observation: There are always at least 2 edge-disjoint paths P_1 and P_2 with $b(P_1) = b(P_2) = b(P)$, where P is the winning path.



Path Markets



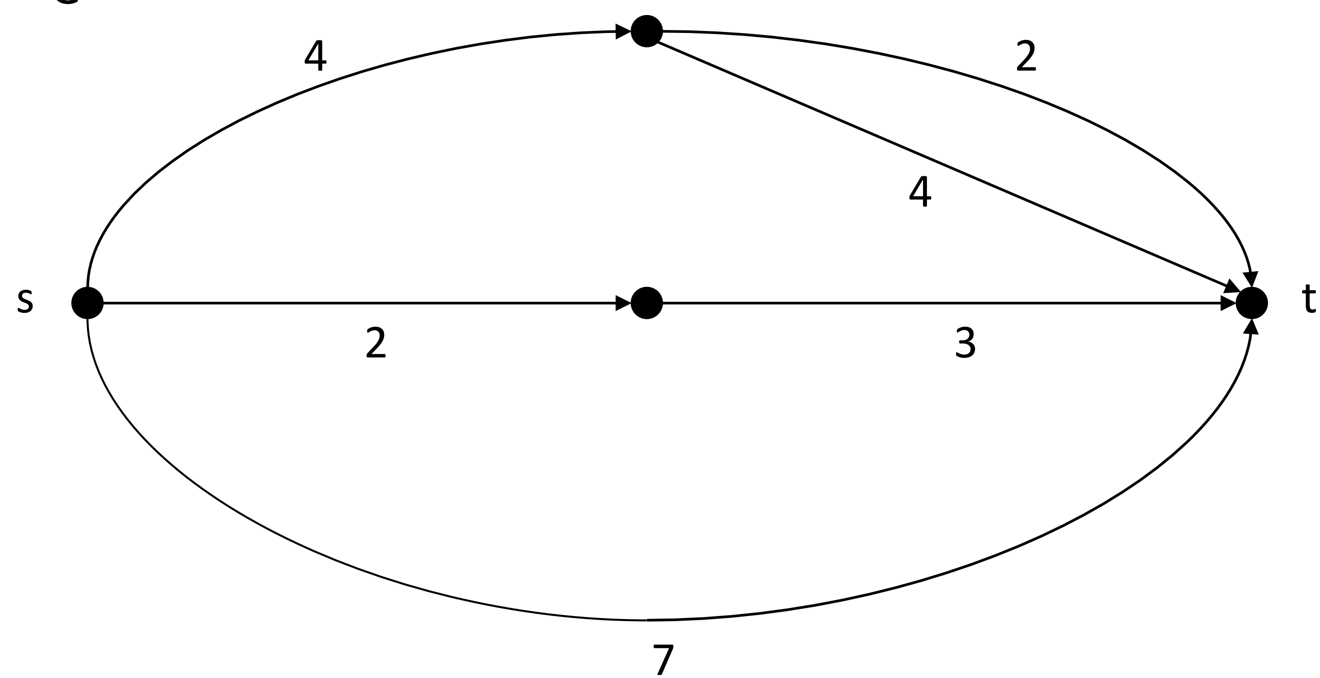
- Proof idea:
  - There always are “tight paths” (tight sets)

    For any e ∈ S, there is another winning feasible set S’ ∈ F with e ∉ S’ and b(S) = b(S’)

  - Any prefix of a tight path is optimal (otherwise the winning path would not be winning).
  - The union of all tight paths only contains optimal s-t paths and is two-connected.

Path Markets

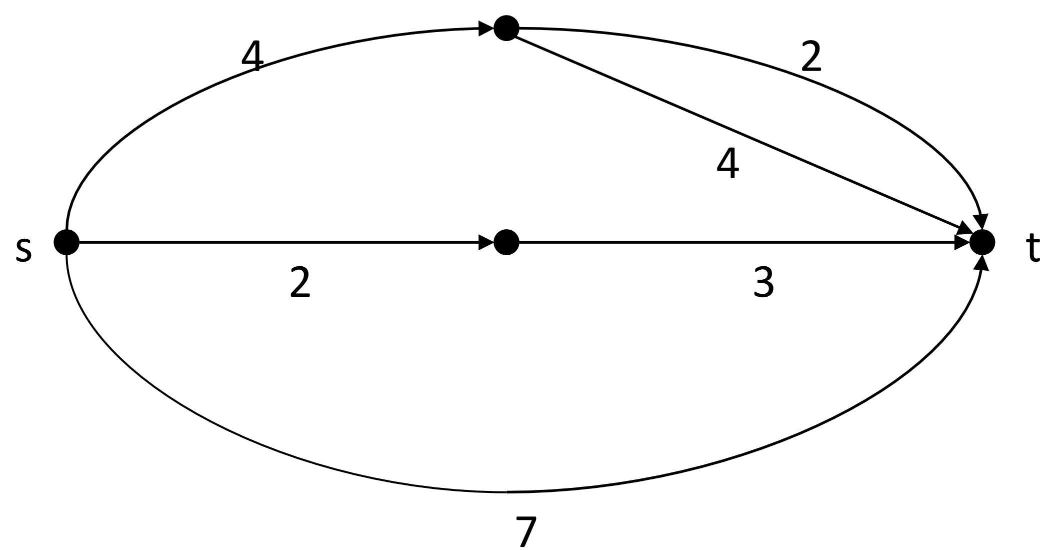
- Proposition:
  - Pick the two cheapest paths by cost, P_1 and P_2.
  - $\min_{G' \subseteq G} p_{G'} = \max\{c(P_1), c(P_2)\}$
    (p_G := total price of G)



Path Markets

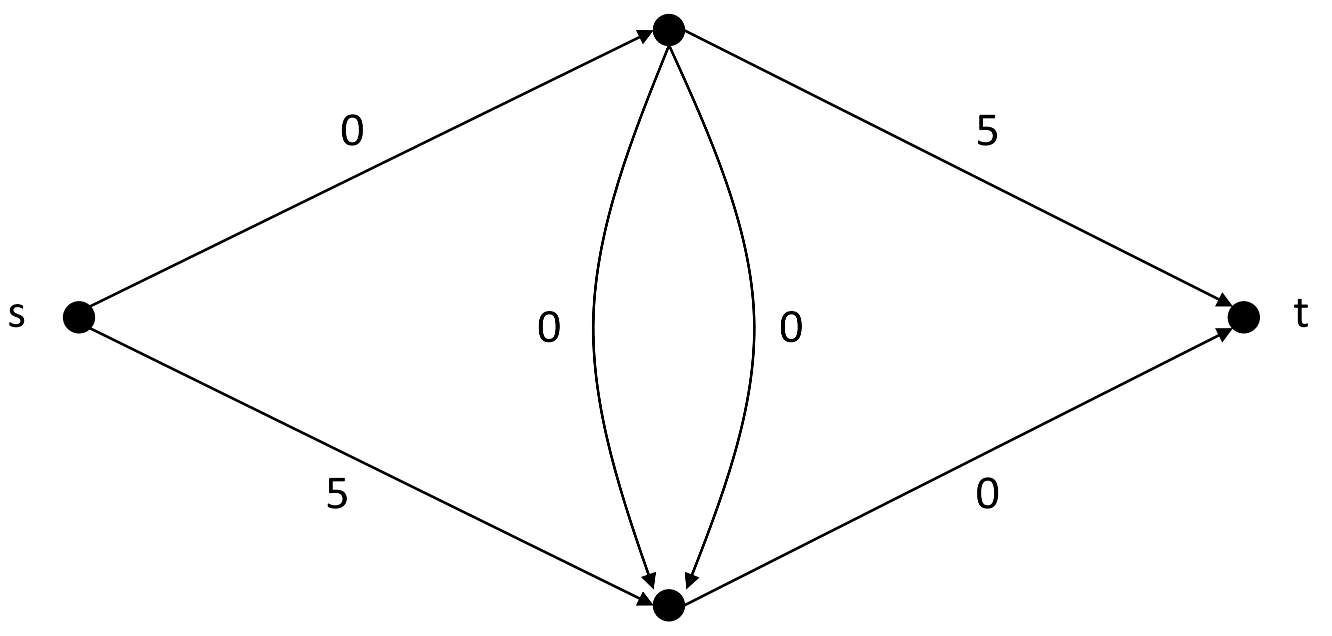
- Now observe that for the two cheapest paths by cost, P1 and P2, $c(P_1) + c(P_2)$ gives an upper bound for p_G.
- Thus, $p_G \le c(P_1) + c(P_2) \le 2 \cdot \max\{c(P_1), c(P_2)\} = 2 \cdot p_{G*}$

→ **The cheap labor cost for path markets is at most 2.**



Path Markets

- This bound is tight:



Conclusion

- Short paper stuffed with proofs
- Exhaustive study of “cheap labor cost” for non-cooperative markets
  - General upper bound |S|
  - Values for common market types

Thanks for your attention!

Questions?