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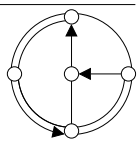

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Principles of Distributed Computing
Exercise 10: Sample Solution

1 Pancake Networks

Generally, observe that $N = |V(P_n)| = n! \in O(n^n) \Rightarrow n \in O(\frac{\log N}{\log \log N})$.

a) See Figure 1. For drawing P_n, first draw n copies of P_{n-1}, each of which will have some $j \in [n]$ fixed as the last vertex. Then there are $(n-2)!$ nodes of such a P_{n-1} connected to the same $(n-1)$-dimensional pancake. To see this, fix v_1 and v_n, the remaining node combinations in the middle will be the link between pancake $P_{n-1}|v_n$ and $P_{n-1}|v_1$. There are $n-1$ such sets in $P_{n-1}|v_n$, each connecting with another $(n-1)$-dimensional pancake.

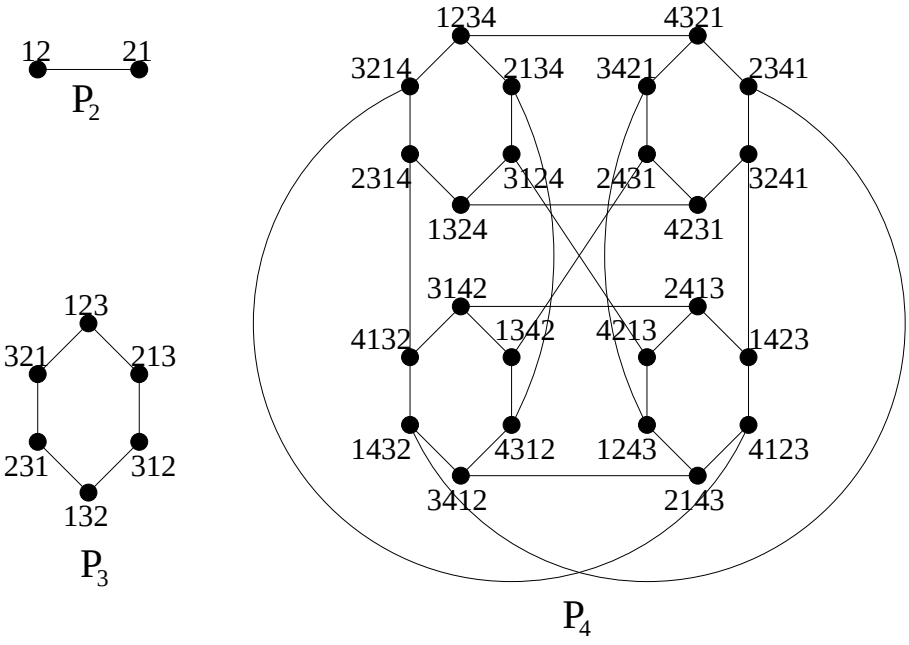

Figure 1: Pancake graphs for $n = 2, 3, 4$.

b) Let us look at the second, more intuitive definition (Eq. (3)). Basically, it states that for every node, there exists exactly one edge for every distinct prefix reversal. So the node degree of P_n can be stated as follows: how many non-trivial prefix reversals are there for a sequence of n nodes? Answer: $n-1$ with edges $e_2, \ldots, e_n$. Succinctly,

$$deg(v) = n - 1 \qquad \forall v \in V(P_n).$$

Thus the degree of an N-node pancake graph is in $O(\log N / \log \log N)$.

c) To give an upper bound on the diameter, we need to determine in how many steps, at most, we can go from one node to any other node. Say we want to get from node $v = v_1 v_2 \dots v_n$ to node $w = w_1 w_2 \dots w_n$. As with all hypercube-like graphs, we will proceed by correcting one "coordinate" at a time. In this case, we start at the back. Since the nodes are all permutations, there will exist a v_j such that $v_j = w_n$. Now take the edges $v \to e_j \to e_n$ to get to node $v^{(1)} = v_N \dots v_{j+1} v_1 v_2 \dots v_{j-1} w_n$. We can relable the indices of $v^{(1)}$ to go again from 1 to $n-1$, leave w_n fixed, find the index j with $v_j = w_{n-1}$, and take the edges $v^{(1)} \to e_j \to e_{n-1}$. Thus, by induction, we need at most 2 edges per correct target index, and we are done after $n-1$ steps. Therefore,

$$D(P_n) \leq 2(n-1)$$

that is, the diameter of P_n is in $O(\log N / \log \log N)$.

Gates and Papadimitriou [1] have also shown that this is asymptotically optimal, that is,

$$D(P_n) \geq n.$$

d) To show that P_n is Hamiltonian, we proceed by induction on n. We will actually show the following stronger claim: In P_n, there exists a Hamiltonian path from $12 \dots (n-1)n$ to $n(n-1) \dots 21$ and the cycle is completed by using edge e_n. Observe that since in P_n the graph looks the same from every vertex, this also holds for any given vertex $v_1 v_2 \dots v_n$.

For $n = 3$: by direct observation, we have the path $123 \to 213 \to 312 \to 132 \to 231 \to 321$ and the final edge $321 \to 123$.

Assume that P_{n-1} has such a Hamiltonian path H_{n-1} from $v_1 v_2 \dots v_{n-1}$ to $v_{n-1} \dots v_2 v_1$. Then we can construct a Hamiltonian path in P_n by concatenating the Hamiltonian paths of the n P_{n-1} subgraphs as follows:

$$\begin{gathered}
a_n = 12 \dots (n-1)n \to H_{n-1} \to (n-1) \dots 21n = b_n \\
b_n \to e_n \to a_{n-1} \\
a_{n-1} = n12 \dots (n-2)(n-1) \to H_{n-1} \to n-2 \dots 1n(n-1) = b_{n-1} \\
b_{n-1} \to e_n \to a_{n-2} \\
\vdots \\
a_2 = 3 \dots (n-1)n12 \to H_{n-1} \to 1n(n-1) \dots 32 = b_2 \\
b_2 \to e_n \to a_1 \\
a_1 = 2 \dots (n-1)n1 \to H_{n-1} \to n(n-1) \dots 21 = b_1
\end{gathered}$$

and we complete the cycle with the final $b_1 \to a_n$ edge. Or, more formally, set

$$\begin{aligned}
a_i &= (i+1)(i+2) \dots n1 \dots (i-1)i \\
b_i &= (i-1) \dots 1n \dots (i+2)(i+1)i
\end{aligned}$$

using $n+1 = 1$ and $1-1 = n$. Then, since the nth coordinate is fixed, the Hamilitonian path H_{n-1} from a_i to b_i is completely contained in $n-1$ dimensions. Its existence is guaranteed by the induction hypothesis. Thus, the Hamiltonian path in n dimensions is given by

$$a_n \overset{H_{n-1}}{\cdots} b_n \to a_{n-1} \overset{H_{n-1}}{\cdots} b_{n-1} \to a_{n-2} \dots a_2 \overset{H_{n-1}}{\cdots} b_2 \to a_1 \overset{H_{n-1}}{\cdots} b_1 \to a_n$$

where $a_n = 12 \dots (n-1)n$ and $b_1 = n(n-1) \dots 21$ as required in the claim.

e) In distributed hash tables, data items are hashed and the first d bits of their hash codes determine which peers are responsible for them. For example, if the node set is $V = [2]^d$, the first d bits of the hash code can be interpreted as the ID of the node where it is stored. If only a subset of all (2^d) possible IDs is used, we can use a virtual ring to determine where data is stored: Each node v has a link to the node w with the next larger ID in the network and the node with the largest ID is connected to the node with the smallest ID.[1] If the first

[1] If these edges are not part of the edge set E, we could simply add these edges to E, which would only increase the degree of each peer by at most 2.

d bits of the hash of a data item are in the range $[v, w)$, then v is responsible for this data item.[2]

If we want to store data on the pancake graph P_n, we can use the fact that P_n is Hamiltonian! Thus, if we use, e.g., the Hamiltonian path from $12 \dots (n-1)n$ to $n(n-1) \dots 21$ constructed in d) as the ring, a unique peer can be determined just like for hypercubic networks. A file is then looked up by simply routing on the pancake to the responsible node.

References

[1] W. H. Gates, C. H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Math. 27, (1979), 47–57.

[2] For example, the hash code 1010 is in the range $[1001, 1110)$. Thus, the node with the ID 1001 is responsible for the corresponding data item.